SOME GLOBAL PROPERTIES OF SINGULARITIES

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A thesis submitted for the degree of Ph.D. at the University of Warwick.

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March 1980.

Contents

Summary.

We use the techniques of algebraic topology to study singularities. We first generalise the Thom polynomial theorem to apply to parametrised maps. We then give a detailed study of the singularities of parametrised real valued functions, with applications to the extension of locally stable maps and Lagrangian immersions. Secondly we use the geometric interpretation of the cohomology theory associated to the Thom spectrum MO given by Quillen, to represent a given singularity set by a cobordism class. We prove a theorem analogous to the Thom polynomial theorem valid for unoriented cobordism theory.

Introduction.

Up to now singularity theory has been mainly local, although there do exist some scattered global results. What is needed is a systematic treatment of the global theory. The objective of global singularity theory is to relate the global topology of the manifolds with the local topology of the singularities involved. In particular we wish to investigate;

(i) Which manifolds can occur as the set of singularities of a given type.

(ii) What are the algebraic constraints on such manifolds.

(iii) How can individual singularities combine together.

(iv) How must certain singularities combine together, due to the global constraints.

Let us review a few of the more striking results. Whitney [32] proved that a generic map from the projective plane to the plane has an odd number of cusps. Pontryagin ([21] Theorem 5 page 269) demonstrated a relationship between characteristic classes and singularities of maps. To Thom these results suggested a far more general theorem. The Thom polynomial theorem (Haefliger and Kosinski [8]) relates the homology class representing the closure of the set of singularities of a given type, to a universal polynomial evaluated on the Stiefel-Whitney classes of the manifolds. Haefliger [9] proved that for a stable map from the torus to the plane there must be at least two cusps, if the fold curve is connected. Singularities can sometimes be eliminated, Ěliasberg and Levine ([14] page 160) have shown that any stable map from a compact manifold N into an orientable 2-manifold is homotopic to a stable map with 0 or 1 cusps according to the parity of the Euler characteristic of N.

In this thesis we use algebraic topology as our main tool to further the global study of singularities. In particular we generalise the Thom polynomial theorem in several ways. We show that the theorem can be applied to parametrised maps. We also associate a cobordism class to a singularity set - again this is inspired by Thom ([27] page 80) - and prove a theorem analogous to the Thom polynomial theorem in unoriented cobordism theory.

Morse theory [18] is the study of generic real valued functions on manifolds; the singularities determine a rich topological structure. The generalisation to parametrised real valued functions is elementary catastrophe theory. Thom [28] suggests many applications of the singularities of parametrised families of functions and other singularities. Motivated by these we develop in some detail the global constraints of a parametrised system.

In Chapter one we generalise the Thom polynomial theorem in two ways, first so that it applies to parametrised maps and second by refining the hypothesis so as not to demand such strict behaviour with respect to a stratification. We then give some examples and in Chapter two specialise to the case of parametrised real valued functions. This is elementary catastrophe theory, which, so far, has been a local classification of the singularities. We consider the homology classes represented by particular singularity sets. In Chapter three using characteristic classes we consider the constraints imposed on the various manifolds that occur in a parametrised system; we prove that a catastrophe map is orientation true and this leads us to an analysis of the degree of the map. In Chapter four we consider the extension of locally stable maps initiated by Markus [15] and Zeeman [33]. We show that certain boundary conditions imply the existence of singularities. An obstruction theorem for Lagrangian immersions is proved in Chapter five. A consequence of this theorem is that the projective plane cannot occur as a Lagrangian immersion in $T^* \mathbb{R}^2$. In the Appendix an explicit Lagrangian immersion of the Klein bottle in $T^* \mathbb{R}^2$ is given.

So far we have converted global properties of singularities into homological properties. We now apply unoriented cobordism theory to the study of singularities. For this we use the geometric interpretation of the cohomology theory associated to the Thom spectrum MO given by Quillen [24]. In Chapter six we prove a theorem analogous to the Thom polynomial theorem for cobordism theory, associating a cobordism class with a singularity manifold.

Ronga [25] and Porteous [22] have given an explicit desingularisation of the first order Thom-Boardman singularities. Using this desingularisation we associate a cobordism class with each of these singularity types. In fact cobordism information can be obtained from all the geometric constructions of Ronga [25] for $\Sigma^{i,j}$ and of Porteous [22],[23] for computing Thom polynomials. Thus, as an example we prove that for a generic map from 4-dimensional projective space to $\mathbb{R}^3$; the 2-dimensional singular manifold is, as an abstract manifold <u>not</u> the boundary of a compact 3-dimensional manifold. In this case the Thom polynomial does not even detect the singularity set.

Applications of our methods to parametrised real valued functions are then given in Chapter eight.

Finally in Chapter nine we give the first results on the image of the singular set. Using stable cobordism and cohomology operations we make some computations. For example, if $f: X^n \to Y^{n+r}$ with $r \geqslant 0$ is a generic map between two compact manifolds and the total singularity set is a manifold, then as an abstract manifold it is the boundary of some compact manifold.

§1 The Thom polynomial theorem for parametrised maps.

All manifolds considered will be smooth and paracompact. Similarly all maps will be C^{∞} .

Throughout we use singular cohomology with coefficients in the field Z_2 and its dual homology theory, singular homology with closed supports. For any manifold V there exists a Poincaré duality isomorphism

$$D : H_*(V) \to H^*(V) .$$

Singular homology with closed supports of a space N is based on chains consisting of infinite linear combinations of singular simplices, such that for all compact sets $K \subset N$ only a finite number of simplices meet K. This homology can also be defined as the inverse limit of the ordinary singular homology groups $H_k(N, N-K; Z_2)$ over the compact subsets K of N , with respect to the inclusion maps.

Definition.

Let P be a submanifold of dimension n in a space Q. Let $\overline{P}$ denote the closure of P in Q. $\overline{P}$ is said to carry a fundamental class if

$$\exists\ \alpha \in H_n(\overline{P}) \text{ such that } \forall\ p \in P$$

$$j_*\alpha \in H_n(\overline{P}, \overline{P}-p) \cong Z_2 \quad \text{is the generator.}$$

Where j is the inclusion $(\overline{P}, \phi\ ) \to (\overline{P}, \overline{P}-p)$.
We write the fundamental class α of $\overline{P}$ as $[\overline{P}]$.

If a fundamental class exists it is unique (see Borel and Haefliger [5] Proposition 2.3).

The following key lemma enables us to pullback fundamental classes.

Lemma 1.

Consider $g: V \to W$ V,W manifolds. $P \to W$ a submanifold such that $\overline{P}$ carries a fundamental class with $g \pitchfork P$ and $g^{-1}(P)$ is a dense subset of $g^{-1}(\overline{P})$.

Then

$\overline{g^{-1}(P)} = g^{-1}(\overline{P})$ carries a fundamental class such that

$$Dj_* [g^{-1}(\overline{P})] = g^* Dk_* [\overline{P}]$$

where

$j: g^{-1}(\overline{P}) \to V$ and $k: \overline{P} \to W$ are inclusions.

Proof.

See Borel and Haefliger ([5] Proposition 2.15)

Let X^n, C^r and Y^m be connected manifolds of dimension n,r and m respectively. Let $F: X \times C \to Y$ be a parametrised family of maps $F_c: X \to Y$, where $F_c(x) = F(x,c)$. To prove a generalisation of the Thom polynomial theorem we first define the jet bundle $J^k(X,Y)$.

Definition.

The Jet bundle $J^k(X,Y)$ is a fibre bundle over $X \times Y$, with fibre the space of jets of order k $\mathbb{R}^n,o \to \mathbb{R}^m,o$ and group $\mathcal{G}^k_n \times \mathcal{G}^k_m$, where $\mathcal{G}^k_p$ is the group of invertible jets of order k $\mathbb{R}^p,o \to \mathbb{R}^p,o$. $\mathcal{G}^k_n \times \mathcal{G}^k_m$ acts by composition on the right and left. Each map $g: X \to Y$ gives rise to a canonically defined map $g^k: X \to J^k(X,Y)$.

Let $F: X \times C \to Y$ be a parametrised family of maps. Each map $F_c: X \to Y$ induces a map

$$F^k_c : X \to J^k(X,Y)$$

This defines a map

$$\tilde{F}^k\colon X \times C \to J^k(X,Y)$$
$$(x,c) \to F_c^k(x)$$

Definition.

Let $E \to X$ be a fibre bundle with fibre E_o and structure group G. Define a type Σ_o in the fibre E_o to be a submanifold, invariant under G, whose closure carries a fundamental class. Since Σ_o is invariant, it determines a unique submanifold Σ_x in the fibre E_x above each point $x \epsilon X$ and hence a subbundle Σ of E. We call Σ a type in the bundle E. A result of Borel and Haefliger ([5] Corollary 2.12) implies that the closure of Σ carries a fundamental class.

Definition.

Let Σ be a type in $J^k(X,Y)$. Given $F\colon X \times C \to Y$, we say F has a singularity of type Σ at (x,c) if $\tilde{F}^k(x,c) \epsilon \Sigma$. We write $\Sigma(F)$ for the set of singularities of type Σ . If $\tilde{F}^k$ is also transverse to Σ at (x,c) we say F has a non-removable singularity of type Σ at (x,c).

In order to apply Lemma 1 we make the following definition.

Definition.

Let Σ be a type in $J^k(X,Y)$. A map $F\colon X \times C \to Y$ is type transverse to Σ if the induced map $\tilde{F}^k$ is transverse to Σ and $(\tilde{F}^k)^{-1}(\Sigma)$ is a dense subset of $(\tilde{F}^k)^{-1}(\overline{\Sigma})$.

Theorem 1.

Let Σ be a type in $J^k(X,Y)$, and let $F\colon X^n \times C^r \to Y^m$ be a parametrised map type transverse to Σ.

Then the closure of the singularities of type Σ carries a fundamental class.

The Poincaré dual after inclusion of this fundamental class in the homology of $X\times C$ is equal to a universal polynomial evaluated on $p^* w_i(X)$ and $F^* w_j(Y)$. Where $w_i(X)$ and $w_j(Y)$ are the Stiefel-Whitney classes of X and Y respectively and p is the projection $X\times C \to X$.

This polynomial depends only on Σ and not on the manifolds X and Y.

<u>Proof.</u>

Consider the following commutative diagram:

$$\begin{tikzcd}
 & & J^k(X,Y) \arrow[d, "\Pi"] & \overline{\Sigma} \arrow[l, "\ell"'] \\
\overline{\Sigma(F)} \arrow[r, "j"] & X\times C \arrow[r, "\tilde{F}^0"] \arrow[ur, "\tilde{F}^k"] & X\times Y &
\end{tikzcd}$$

Haefliger and Kosinski [8] prove that there exists a polynomial P_Σ such that

$$\Pi^{*-1} D\ell_* [\,\overline{\Sigma}\,] = P_\Sigma(p_1^* w_i(x),\ p_2^* w_j(Y))$$

where $p_1\colon X\times Y \to X$ and $p_2\colon X\times Y \to Y$ are projections and P_Σ depends only on Σ.

By hypothesis F is type transverse to Σ and so by Lemma 1

$$Dj_* [\,\overline{\Sigma(F)}\,] = (\tilde{F}^k)^* D\ell_* [\,\overline{\Sigma}\,].$$

By the commutative diagram

$$\begin{aligned} Dj_* [\,\overline{\Sigma(F)}\,] &= (\tilde{F}^0)^* \Pi^{*-1} D\ell_* [\,\overline{\Sigma}\,] \\ &= (\tilde{F}^0)^* P_\Sigma (p_1^* w_i(X),\ p_2^* w_j(Y)) \end{aligned}$$

$$= P_\Sigma(P^* w_i(X),\ F^* w_j(Y)).$$

Corollary 1 (Haefliger and Kosinski [8]).

Let Σ be a type in $J^k(X,Y)$ and $f: X^n \to Y^m$ a map type transverse to Σ. Then the closure of the singularities of type Σ carries a fundamental class.

Let $j: \overline{\Sigma(f)} \to X$ be the inclusion, then there exists a universal polynomial such that

$$Dj_* [\overline{\Sigma(f)}] = P_\Sigma(w_i(X),\ f^* w_j(Y)).$$

Proof.

Let the parameter manifold C in the theorem be a single point.

In particular the Thom polynomials for ordinary mappings and parametrised mappings are essentially the same, and so computations of these universal polynomials by Porteous [22] [23] , Ronga [25] and Sergeraert [26] apply to parametrised maps.

Corollary 2.

Let $F: X^n \times C^r \to Y^m$ be type transverse to Σ, where Σ is a type of codimension $> n$ in $J^k(X,Y)$. If $w_i(Y)=0 \ \forall \ i > 0$ then the fundamental class of $\overline{\Sigma(F)}$ is killed by inclusion in the homology of $X \times C$.

Proof

By the theorem if suffices to show that the polynomial $P_\Sigma(p^* w_i(X),$ $F^* w_j(Y)) = 0$. This polynomial reduces to a polynomial of degree $> n$ in $p^* w_i(X)$. Since the dimension of X is n this polynomial must be zero.

The above corollary shows that for particular target manifolds such as $Y = \mathbb{R}^m$, singularities which do not occur transversely for maps $X \to Y$ but which do occur transversely for parametrised maps $X \times C \to Y$ have zero Thom polynomials. The following example shows that this is not always the case. Specifically we consider a type Σ in $J^k(X,Y)$ of codimension greater than the dimension of X, and construct a map $F: X \times C \to Y$ type transverse to Σ such that the closure of the singularities of type Σ are not killed by inclusion in the homology of $X \times C$.

Example.

Let $X = S^1$, $C = P^2$ and $Y = P^2$, where S^1 and P^2 are the circle and projective plane respectively. The Stiefel-Whitney classes of S^1 and P^2 are well known, $w_i(S^1) = 0$ for $i > 0$ and $w_i(P^2) = a^i$ for $i > 0$, where a is the generator of $H^1(P^2)$.

Let Σ^1 be the type in $J^1(S^1, P^2)$ corresponding to the zero section, i.e. linear maps of rank zero. Therefore the singularities of type Σ^1 occur transversely with codimension two. We consider a map type transverse to Σ^1. Porteous [22] has computed the ordinary Thom polynomial for the closure of Σ^1, for a map $f: A^s \to B^{s+1}$, it equals the second Stiefel-Whitney class of the difference bundle $f^*TB - TA$. To apply this polynomial to parametrised maps $F: X \times C \to Y$ we just consider the second Stiefel-Whitney class of the difference bundle $F^*TY - p^*TX$.

In our case
$$\begin{aligned} w_2(F^*TP^2 - p^*TS^1) &= w_2(F^*TP^2) \\ &= F^* w_2(P^2) \\ &= F^* a^2 . \end{aligned}$$

This is non-zero if F^* is injective. However, choosing F so that it is homotopic to the projection onto P^2 ensures that F^* is injective.

We recall from Boardman [4] important examples of types in $J^k(X,Y)$ where X and Y have the same dimension. The Thom-Boardman types Σ^1, $\Sigma^{1,1}$ and in general Σ^{1^q}, with q copies of 1, has codimension q in $J^k(X,Y)$. Boardman defines these algebraically and their closure is a real algebraic set and so carries a fundamental class (Borel and Haefliger [5] Theorem 3.7). The Thom polynomials for these types have been computed by Porteous [22] and Sergeraert [26]. The non-removable singularities of type Σ^{1^q} underlie the cuspoid elementary catastrophes of Thom [28], namely the fold, cusp, swallowtail, butterfly,... for q=1,2,3,4,... etc. However, note that the cuspoid elementary catastrophes have more structure than just non-removable singularities of type Σ^{1^q}. Furthermore, the other elementary catastrophes do not arise as non-removable singularities for any type in $J^k(X,Y)$.

§2 The homology of the singular sets of elementary catastrophe theory.

We now restrict attention to parametrised families of real valued functions $F: X^n \times C^r \to \mathbb{R}$. Instead of the jet bundle $J^k(X, \mathbb{R}) \to X\times \mathbb{R}$ defined in the first section we consider the jet bundle $J^k(X) \to X$. This bundle has the same fibre as $J^k(X, \mathbb{R})$, the k-jets of maps $\mathbb{R}^n,o \to \mathbb{R},o$, but has a smaller group acting on these fibres. Instead of the group $\mathcal{G}^k_n \times \mathcal{G}^k_1$ acting on the right and left we consider the group $\mathcal{G}^k_n$ acting on the right. Each map $g: X \to \mathbb{R}$ gives rise to a canonically defined map $g^k: X \to J^k(X)$.

For a map $F: X{\times}C \to \mathbb{R}$ we have an induced map:

$$\tilde{F}^k : X \times C \to J^k(X)$$
$$(x,c) \to F^k_c(x) .$$

We can also define types in $J^k(X)$ as before and also the singularity type of a map $F: X \times C \to \mathbb{R}$ corresponding to a given type.

To define a type it suffices to consider a single fibre. The fibres of $J^k(X, \mathbb{R})$ and $J^k(X)$ are the same, however, the group action is different, so although a type in $J^k(X, \mathbb{R})$ is a type in $J^k(X)$ the converse is not necessarily true. In particular Theorem 1 and Corollary 2 do not have immediate application. However, we can deduce the following results.

Theorem 2.

Let I be the type in $J^k(X)$ induced by the Morse functions. Let $F: X{\times}C \to \bar{\mathbb{R}}$ be a parametrised family of maps type transverse to I. Then the closure of the singularities of type I carries a fundamental class whose Poincaré dual after inclusion in the homology of $X{\times}C$ is $p^* w_n(X)$. Where $w_n(X)$ is the top Stiefel-Whitney class of X and $p{:}X{\times}C \to X$ projection.

Proof.

I is also a type in $J^k(X, \mathbb{R})$ so we can apply Theorem 1. We need to consider the Thom polynomial for singular maps $X^n \to \mathbb{R}$. This polynomial has been computed by Porteous [22] as $w_n(X)$, where $w_n(X)$ is the top Stiefel-Whitney class of X. It also follows easily from Morse theory. Applying Theorem 1 we obtain $p^* w_n(X)$.

Example.

Let S^1 and S^2 denote the circle and the 2-sphere respectively. Consider a parametrised family of maps $F: S^2\times S^1 \to \mathbb{R}$ type transverse to I, the type induced by the Morse functions. The closure of the singularities of type I forms a one dimensional submanifold of $S^2\times S^1$. This manifold is compact and so consists of a finite number of embedded circles. The Euler characteristic of S^2 is even so $w_2(S^2)=0$. The theorem says that the homology class in $H_1(S^2\times S^1)$ represented by the sum of these circles is zero. Note that an individual circle may wind around S^1 with arbitrary degree.

Theorem 3.

Let $F: X\times C \to \mathbb{R}$ be a parametrised family of maps and Σ a type in $J^k(X)$ of codimension $s > n$, where n is the dimension of X. If F is type transverse to Σ then the closure of the set of singularities of type Σ carries a fundamental class. This homology class is killed by inclusion in the homology of $X\times C$.

Proof.

Consider the following diagram

$$\begin{tikzcd}
 & & J^k(X) \arrow[d, "\Pi"] & \overline{\Sigma} \arrow[l, "\ell"'] \\
\overline{\Sigma(F)} \arrow[r, "j"] & X\times C \arrow[ur, "\tilde{F}^k"] \arrow[r, "p"] & X &
\end{tikzcd}$$

where j and ℓ are inclusions.

By Lemma 1

$$Dj_* [\overline{\Sigma(F)}] = \tilde{F}^{k*} D\ell_* [\overline{\Sigma}]$$

$$= p^* \pi^{*-1} D\ell_* [\overline{\Sigma}]$$

by the commutative diagram.

Since Σ has codimension $s > n$ in $J^k(X)$ we have

$$D\ell_* [\overline{\Sigma}] \in H^s(J^k(X)).$$

Hence $\pi^{*-1} D\ell_* [\overline{\Sigma}] \in H^s(X)$

But the dimension of X is n so $H^s(X) = 0$.

It follows that $Dj_* [\overline{\Sigma(F)}] = 0$

and $j_*[\overline{\Sigma(F)}] = 0$ as claimed.

We now consider the space of all maps $F\colon X^n \times C^r \to \mathbb{R}$ with the Whitney C^∞- topology. To obtain interesting structure we need to restrict attention to a generic set of maps. For $r \leq 5$ Thom's Theorem (Trotman and Zeeman [30]) says there exists an open dense set of maps such that the induced map $\tilde{F}$ is transverse to all the orbits of $J^k(X)$. The orbits are finite in number and are classified by Trotman and Zeeman in [30] (see 4.1). For $r > 6$ there are an infinite number of orbits of codimension $n+6$ in $J^k(X)$, providing n=dimension of X is greater than two. So we cannot hope for $\tilde{F}$ to be transverse to all the orbits. However, for our purposes it suffices to have $\tilde{F}$ transverse to the Morse orbits, so we make the following definition to obtain a generic set of maps.

Definition.

A map $F: X^n \times C^r \to \mathbb{R}$ is good if $\tilde{F}$ is type transverse to the type induced by the Morse functions.

A Morse singularity can be assigned an index and its 2-jet is of maximal rank. These two invariants partition the orbits in $J^k(X)$.

Definition.

Let $\eta \varepsilon J^k(X)$

If $j^1\eta = 0$ $j^2\eta$ is a quadratic form Q in n variables.

Define rank η = rank Q

index η = index Q = number of positive eigenvalues of Q.

The terms index and rank can also be applied to orbits.

The Morse orbits are completely classified by index. The orbits of corank 1 have also been completely classified, see [2] and [3] . The orbit $\pm A^i_j$ has index i and codimension $n+j-1$ in $J^k(X)$. Arnold [2] combines all these orbits for fixed j, to form A_j, since he is working over the complex numbers and so the index is not an invariant. The class - ification of orbits of corank 2 or greater is incomplete, see Arnold [3] .

Definition.

Let $I^k(X)$ be defined by the short exact sequence of bundles over X.

$$0 \to I^k(X) \to J^k(X) \xrightarrow{\sigma} T^*X \to 0$$

where σ is the projection onto $J^1(X)$, the cotangent bundle.

$I^k(X)$ inherits the same group as $J^k(X)$. If Σ is a type of codimension $\geq n$ in $J^k(X)$ then we use the same symbol for the type in $I^k(X)$ which is the restriction of Σ to $I^k(X)$.

Let $F\colon X^n \times C^r \to \mathbb{R}$ be a good map. We define the <u>critical manifold</u> $M = \{(x,c) \mid dF_c(x) = 0\} = (\hat{F}^k)^{-1}\, I^k(X)$. M is empty or is an r-dimensional submanifold of $X \times C$. Let i denote the inclusion of M into $X \times C$.

$$\begin{tikzcd} M \arrow[r, "i"] \arrow[dr, "\chi_F"'] & X \times C \arrow[d] \\ & C \end{tikzcd}$$

Let $\chi_F\colon M \to C$ be the map induced by composing the inclusion into $X \times C$ with projection onto the second factor.

We call χ_F the <u>catastrophe map</u>, X the <u>state space</u> and C the <u>parameter space</u>.

Since $M = (\hat{F}^k)^{-1}\, I^k(X)$ we can factor $\hat{F}$ through $I^k(X)$ by the following commutative diagram.

$$\begin{tikzcd} M \arrow[r, "i"] \arrow[d, "\hat{F}^k"'] & X \times C \arrow[d, "\hat{F}^k"] \\ I^k(X) \arrow[r] & J^k(X) \end{tikzcd}$$

We obtain a map $\hat{F}^k\colon M \to I^k(X)$.

For convenience of notation we will write $\hat{F}$ for $\hat{F}^k$, similarly we may write χ for χ_F.

<u>Lemma 2</u>.

Let A,B,C and D be manifolds such that

$$\begin{array}{lll} A \xrightarrow{f} B & & \\ \qquad\quad \cup & \text{with } f \pitchfork C & \text{and } f \pitchfork D \\ \qquad\quad C & & \\ \qquad\quad \cup & \text{Let } f' = f|_{f^{-1}C} & \text{then } f' \pitchfork D. \\ \qquad\quad D & & \end{array}$$

Proof.

The lemma follows from elementary transversality theory and is left to the reader.

Lemma 3.

Let $F: X \times C \to \mathbb{R}$ be a good map and Σ a type in $I^k(X)$.

$\tilde{F}^k$ is type transverse to Σ in $J^k(X) \Leftrightarrow \hat{F}$ is type transverse to Σ in $I^k(X)$.

Proof.

Transversality follows immediately from the previous lemma with $f = \tilde{F}^k$, $f' = \hat{F}$ and $D = \Sigma$. The condition on the closure of Σ for $\hat{F}$ is automatically satisfied if it is true for $\tilde{F}^k$, and conversely.

Let Σ be a type in $I^k(X)$. We say χ has a singularity of type Σ at m if $\hat{F}(m) \in \Sigma$. Lemma 3 implies that χ has a singularity of type Σ at m if and only if F has a singularity of type Σ at m. Let $\Sigma(\chi) = \Sigma(F) = \{m \in M \mid \hat{F}(m) \in \Sigma\}$.

If $\hat{F}$ maps m into an orbit of corank q then the projection χ at m drops rank q. Let $\hat{F}(m) \in \pm A^i_j$ for some i and j, the rank of $\pm A^i_j$ is $n-1$, so the projection of χ at m drops rank 1. If $\hat{F}$ is transverse to the orbit $\pm A^i_j$ at m then the projection gives a cuspoid and so we sometimes abuse notation and call the orbits $\pm A^i_j$ cuspoids. Similarly the orbits of corank two are sometimes called umbilics. From the point of view of pure mathematics $\hat{F}$ is more useful than χ, since for a residual set of maps F can be made type transverse to any type in $I^k(X)$, whereas a residual set of maps χ cannot be made type transverse to types in $J^k(M,C)$, other than the types Σ^{1^q}. (See the end of the last chapter). However for applications the map χ and the elementary catastrophes are more important.

In Theorem 3 we proved that if Σ is a type in $I^k(X)$ of positive codimension, and the map $F: X\times C \to \mathbb{R}$ is type transverse to Σ, then the fundamental class $[\,\overline{\Sigma(F)}\,]$ is killed by inclusion in the homology of $X\times C$. We now prove a lemma in order to deduce a stronger result, namely that it is already killed in the critical manifold.

<u>Lemma 4</u>.

Let Σ be a type in $I^k(X)$ of positive codimension. Then there exists a section τ of $I^k(X)$ such that $\tau(X) \cap \overline{\Sigma} = \phi$.

<u>Proof</u>.

Let $HOM(\circ^2 TX, \mathbb{R})$ denote the bundle over X whose fibre is the symmetric bilinear maps $\mathbb{R}^n\times \mathbb{R}^n \to \mathbb{R}$. Let $j: HOM(\circ^2 TX, \mathbb{R}) \to I^k(X)$ denote the inclusion. X can be given a Riemannian structure, that is a section s of the bundle $HOM(\circ^2 TX, \mathbb{R})$, $s(x)$ is in particular a non-singular bilinear map. Define $\tau = j\circ s$, τ has its image in the type in $I^k(X)$ induced by the Morse functions.

<u>Theorem 4</u>.

Let Σ be a type of positive codimension in $I^k(X)$. Let $F: X\times C \to \mathbb{R}$ be a good map type transverse to Σ, then the closure of the singularities of type Σ carries a fundamental class which is killed by inclusion in the homology of M.

<u>Proof</u>.

Consider the following commutative diagram.

$$\begin{tikzcd}
 & & & I^k(X) \arrow[d, "\pi"] & \overline{\Sigma} \arrow[l, "j"'] \\
\overline{\Sigma(F)} \arrow[r, "\ell"] & M \arrow[r, "i"] \arrow[urr, "\hat{F}"] & X\times C \arrow[r, "p"] & X &
\end{tikzcd}$$

j and ℓ are inclusions.

F is type transverse to Σ, so applying Lemma 1 we have

$$D\ell_* \left[\, \overline{\Sigma(F)} \,\right] = \hat{F}^* Dj_* \left[\, \overline{\Sigma} \,\right]$$

$$= i^* p^* \pi^{*-1} Dj_* \left[\, \overline{\Sigma} \,\right]$$

by the commutative diagram.

Let $\tau\colon X \to I^k(X)$ be a section given by the previous lemma, so that $\tau(X) \cap \overline{\Sigma} = \phi$. In particular τ is type transverse to Σ, so applying Lemma 1 we have

$$\tau^* Dj_* [\, \overline{\Sigma} \,] = 0$$

But τ is a section so $\tau^* = \pi^{*-1}$.

Therefore we have $D\ell_* \left[\, \overline{\Sigma(F)} \,\right] = 0$

and $\ell_* \left[\, \overline{\Sigma(F)} \,\right] = 0.$

When X is 1-dimensional we can extend the above method, in this case $J^k(X)$ is a Whitney sum of line bundles. $J^k(X) = \bigoplus_{p=1}^{k} \mathrm{HOM}(\odot^p TX, \mathbb{R})$, where $\mathrm{HOM}(\odot^p TX, \mathbb{R})$ is the line bundle of symmetric p-linear maps $TX \to \mathbb{R}$. Chopping off the first bundle $\mathrm{HOM}(TX, \mathbb{R})$ we obtained $I^k(X)$ ($= I_1^k(X)$ say). Chopping off the first j bundles we obtain $I_j^k(X)$. That is, we define $I_j^k(X)$ inductively by the following short exact sequence of bundles over X

$$0 \to I^k_{p+1}(X) \to I^k_p(X) \xrightarrow{\text{proj}} \mathrm{HOM}(\mathbb{O}^p TX, \mathbb{R}) \to 0$$

Each bundle $I^k_j(X)$ is a trivial bundle over X, since the tangent bundles over $\mathbb{R}$ and the circle are trivial. So each bundle has a section s_j which does not meet the zero section.

We need to decide which manifolds are types in $J^k(X)$. The right equivalence orbits in a fibre of $J^k(X)$ are known, they have representatives

$$x \to \pm x^p \quad \text{for } p \text{ even}$$

$$\text{and } x \to x^p \quad \text{for } p \text{ odd.}$$

With p even, the closure of a single orbit $+x^p$ or $-x^p$ does not carry a fundamental class, since the closure is a manifold with boundary. So for p even, we need to combine the two orbits to obtain a type. Hence there exists one type for each integer. The closure of each type $x \to \pm x^{p+1}$ is the manifold $I^k_p(X)$ and has codimension p in $J^k(X)$.

There exists an open dense set of maps type transverse to all the types in $I^k(X)$. Given a generic map $F: X^1 \times C \to \mathbb{R}$ there exists a sequence of manifolds in $X \times C$

$$M_{r+1} \subset M_r \subset \ldots \qquad \ldots \subset M_1 \subset X{\times}C(=M_0 \text{ say})$$

corresponding to the cuspoids

$$\ldots \subset \text{cusps} \subset \text{folds} \subset \text{Morse sings} \subset X{\times}C\ ,$$

where $M_p = \bar{F}^{-1}(I^k_p(X))$ has codimension p in $X{\times}C$.

<u>Theorem 5.</u>

Let $F: X^1{\times}C^r \to \mathbb{R}$ be a generic map. Then for all $q > 0$, the homology class representing the manifold M_q is killed by inclusion in the homology of M_{q-1} .

Proof

By the transversality Lemma 2 we have a restriction h of $\tilde{F}$ to M_{q-1} and a restriction $\tilde{h}$ of h to M_q with h transverse to $I_q^k(X)$.

$$
\begin{tikzcd}
M_{q-1} \arrow[r, "h"] & I_{q-1}^k(X) \\
M_q \arrow[u, "j"] \arrow[r, "\tilde{h}"] & I_q^k(X) \arrow[u, "\ell"']
\end{tikzcd}
$$

Let j and ℓ denote inclusions.

By Lemma 1 we have

$$Dj_* \left[M_q \right] = h^* D\ell_* \left[I_q^k(X) \right]$$

Consider the following diagram which homotopy commutes

$$
\begin{tikzcd}
M_{q-1} \arrow[r, "h"] \arrow[dr, "p"'] & I_{q-1}^k(X) \\
 & X \arrow[u, "s_{q-1}"']
\end{tikzcd}
$$

Where p is the restriction of the projection $X \times C \to X$.

We have

$$
\begin{aligned}
Dj_* [M_q] &= p^* s_{q-1}^* D\ell_* \left[I_q^k(X) \right] \\
&= 0
\end{aligned}
$$

since s_{q-1} avoids $I_q^k(X)$.

So we have $j_* \left[M_q \right] = 0$.

<u>Example.</u>

Let $F: X^1 \times C^2 \to \mathbb{R}$ be a generic map with X and C compact. Then the manifold representing the closure of the singularities of type x^3(the folds) is one dimensional and consists of a finite number of circles. On each circle the number of singularities of type $\pm x^4$(the cusps) is even. For example, the following picture cannot be the image of the singular set in C.

All the results of this section apply to types in the jet bundle $J^k(X)$. As we have seen, single orbits are not necessarily types. Haefliger ([13] section 4) gives a construction for the smallest real algebraic set containing an orbit and so by a result of Borel and Haefliger ([5] Theorem 3.7) this produces a type. To produce types it suffices to restrict attention to a single fibre of the jet bundle. The fibre is the space of k-jets of maps $\mathbb{R}^n,0 \to \mathbb{R},0$ acted upon by the group $\mathcal{G}_n^k$. Haefliger embeds this fibre in the space of k-jets of analytic maps $\mathbb{C}^n,0 \to \mathbb{C}, 0$ with group $\tilde{\mathcal{G}}_n^k$, the invertible k-jets of analytic maps $\mathbb{C}^n,0 \to \mathbb{C}^n,0$. He then proves that if θ is a real orbit and $\theta\tilde{\mathcal{G}}_n^k$ its complexification, then the closure of $\theta\tilde{\mathcal{G}}_n^k \cap \{k \text{ jets } \mathbb{R}^n,0 \to \mathbb{R},0\}$ is a real algebraic set.

Although Haefliger's construction produces the smallest real algebraic set containing the orbit θ, this is not necessarily the smallest manifold containing θ whose closure carries a fundamental class. We will later show that the closure of the singularities of index j whose 2-jet drops rank by i carries a fundamental class. In general, however, orbits of the same Arnold notation [2], [3] but different indices are combined by Haefliger's construction and cannot be separated. For example, this is the situation with the cusps.

We now prove some lemmas concerning fundamental classes.

<u>Lemma 5.</u>

$$\left.\begin{array}{l} \text{Let } a,b \text{ be } m\text{-manifolds} \\ \bar{a} \cup \bar{b} \text{ carries a fundamental class} \\ \dim(\bar{a} \cap \bar{b}) \leq m-2 \end{array}\right\} \Rightarrow \begin{array}{l} \bar{a} \text{ carries a} \\ \text{fundamental class.} \end{array}$$

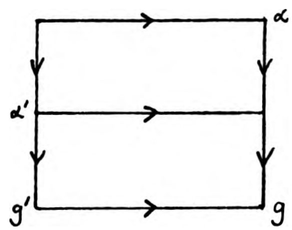
<u>Proof.</u>

Given any $p \in a$ consider the following diagram.

$$\begin{tikzcd}
H_m(\bar{a}) \oplus H_m(\bar{b}) \arrow[r, "\cong"] \arrow[d] & H_m(\bar{a} \cup \bar{b}) \arrow[d] \\
H_m(\bar{a}) \arrow[r, "\cong"] \arrow[d] & H_m(\bar{a} \cup \bar{b}, \bar{b}) \arrow[d] \\
H_m(\bar{a}, \bar{a}-p) \arrow[r, "\cong"] & H_m(\bar{a} \cup \bar{b}, \bar{a} \cup \bar{b}-p)
\end{tikzcd}$$

The top homomorphism occurs in the Mayer-Vietoris sequence of the triad $(\bar{a} \cup \bar{b}, \bar{b}, \bar{a})$, and is an isomorphism since $\dim(\bar{a} \cap \bar{b}) \leq m-2$. The second homomorphism occurs in the Mayer-Vietoris sequence of the pair of triads $(\bar{b}, \bar{b}, \phi) \subset (\bar{a} \cup \bar{b}, \bar{b}, \bar{a})$, and is an isomorphism for the same reason. The upper square commutes because there is an injection from one Mayer-Vietoris sequence to the other. The lower square commutes because it is induced by injection. The bottom homomorphism is an isomorphism by excision.

Let α be the fundamental class of $\bar{a} \cup \bar{b}$. Then α projects to the generator g as shown.



Hence the image α' of the pullback of α projects onto the pullback g' of g which is the generator of $H_m(\bar{a}, \bar{a}\text{-}p)$. Hence $\bar{a}$ carries a fundamental class α'.

□

<u>Lemma 6.</u>

Let a,b be manifolds invariant under the group action of $I^k(X)$ such that

$$\eta \in a \quad \text{rank } \eta = r, \quad \text{index } \eta = i$$

and $$\xi \in b \quad \text{rank } \xi = r, \quad \text{index } \xi < i$$

If $\gamma \in \bar{a} \cap \bar{b}$ then index $\gamma < i$

and rank $\gamma < r$.

<u>Proof.</u>

For each r $\{ \beta |$ index $\beta \leqslant r \}$ is closed.

So $\eta \in \bar{a}$ => index $\eta \leq i$

$\eta \in \bar{b}$ => index $\eta < i$

Let $\gamma \in \bar{a} \cap \bar{b}$ => index $\gamma < i$.

For $-\gamma \in \bar{a}$ dropping index means that the number of essential variables in the quadratic form of $j^2\gamma$ decreases and the rank of γ is therefore less than r.

Let $\Sigma^{n,q}$ denote the type in $I^k(X^n)$ consisting of those k-jets whose 2-jet drops rank by q. The closure of $\Sigma^{n,q}$ consists of the types $\Sigma^{n,q+i}$ for $i \geq 0$ $\Sigma^{n,q+1}$ has codimension q+1 in $\Sigma^{n,q}$, therefore $\overline{\Sigma}^{n,q}$ carries a fundamental class.

<u>Theorem 6.</u>

The subset of $\Sigma^{n,q}$ of index p is a type in $I^k(X^n)$.

<u>Proof.</u>

$\overline{\Sigma}^{n,q}$ carries a fundamental class. Let a_k denote the submanifold of $\Sigma^{n,q}$ of index k.

Let i be the largest index so that we can write

$$\overline{\Sigma}^{n,q} = \cup \overline{a}_k = \overline{a}_i \cup (\bigcup_{j<i} \overline{a}_j).$$

Let $\gamma \in \overline{a}_i \cap (\bigcup_{j<i} \overline{a}_j)$ then rank $\gamma < q$ by Lemma 6 . Therefore $\dim(\overline{a}_i \cap (\bigcup_{j<i} \overline{a}_j)) \leq \dim \Sigma^{n,q} - 2$

Hence by Lemma 5 $\overline{a_i}$ carries a fundamental class, by the symmetry of Lemma 5 $\cup \overline{a_j}$ does also. By repeating this argument we can show that $\overline{a_p}$ carries a fundamental class for each p.

□

<u>Examples.</u>

The A_2^i orbits of each index i are types, corresponding to fold catastrophes of each index. Although our methods cannot distinguish between the hyperbolic and elliptic umbilics the theorem proves that we can separate the umbilics with different indices. The closure of $\Sigma^{n,3}$ is the same as the closure of the singularities of Arnold [3] of type P_8, and so these can be distinguished by different indices. The closure of the collection of other unimodular families in the sense of Arnold [3] also carry fundamental classes although we need to consider all the indices to form a type. All the cuspoids of a given codimension greater than one need to be lumped together to form a type. Similarly the umbilics of a fixed codimension greater than one need to be lumped together to form a type. Similarly the umbilics of a fixed codimension greater than three need to be combined. As an example our methods do not distinguish between the parabolic umbilic or the dual parabolic umbilic with any of the $n-1$ possible indices.

§3 Topological restrictions on C, M and X.

In this chapter we give relationships between the topology of the parameter space, state space and critical manifold for parametrised real valued functions. We first give examples of singularity sets that can occur by suspending good maps.

Definition.

Let $F: X \times C \to \mathbb{R}$ be a good map

If D is any manifold we define the suspension F^* of F

$$F^* : X \times C \times D \to \mathbb{R}$$

$$\text{by} \quad F^*(x,c,d) = F(x,c).$$

Lemma 7.

Let $F: X \times C \to \mathbb{R}$ be a good map and $F^*: X \times C \times D \to \mathbb{R}$ its suspension. If Σ is a type in $I^k(X)$ and F is type transverse to Σ then F^* is type transverse to Σ.

Furthermore, $\overline{\Sigma(F^*)} = \overline{\Sigma(F)} \times D$.

Proof.

The transversality properties are immediate.

$$\overline{\Sigma(F)} = \text{Closure} \quad \{(x,c) \mid \hat{F}(x,c) \in \Sigma\}$$

$$\begin{aligned} \overline{\Sigma(F^*)} &= \text{Closure} \quad \{(x,c,d) \mid \hat{F}^*(x,c,d) \in \Sigma\} \\ &= \text{Closure} \ \{(x,c,d) \mid \hat{F}(x,c) \in \Sigma\} \\ &= \overline{\Sigma(F)} \times D. \end{aligned}$$

□

Examples.

1 If X is non-compact there exists a Morse function on X with no critical points (See Hirsch [11] Theorem 4.8), so for all manifolds C

there exist good maps $F: X \times C \to \mathbb{R}$ with empty critical manifold.

2 For all $m > 0$ there exists a manifold X with a Morse function g having m critical points. Then $g^* : X \times C \to \mathbb{R}$ has a critical manifold with on components each diffeomorphic to C. In particular any manifold can occur as a critical manifold.

3 Similarly any manifold of the right codimension can occur as a component of the singularity set of a given type. That is, if $F: X \times C \to \mathbb{R}$ has isolated singularities of type Σ then $F^*: X \times C \times D \to \mathbb{R}$ has the manifold D as components of its singularities of type Σ.

If $\chi : M \to C$ is a catastrophe map this imposes conditions on X, trivially since M must embed in $X \times C$. For example, 3-dimensional projective space p^3 cannot occur as a component of M for any good map $F: \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ since p^3 does not embed in $\mathbb{R}^4$.
Therefore, although we have shown that any manifold can be the critical manifold and of course any connected manifold can be the state or parameter space, there are restrictions relating the topology of C, M and X.
We now consider some of these restrictions.

Definition.

The vector bundles E and F over X are stably equivalent written $E \overset{S}{\approx} F$, if there exists trivial bundles ε^p, ε^q over X such that

$$E \oplus \varepsilon^p = F \oplus \varepsilon^q$$

The ranks of ε^p and ε^q are p and q .

Definition.

Let $F: X \times C \to \mathbb{R}$ be a good map. Let $\hat{M}$ be a connected component of the critical manifold M. We define a restricted catastrophe map $\tilde{\chi} : \hat{M} \to C$ to be the restriction of χ to $\hat{M}$.

<u>Theorem 7.</u>

Let $F\colon X{\times}C \to \mathbb{R}$ be a good map.

Then $TM \overset{S}{\approx} \chi^* TC$ and $T\tilde{M} \overset{S}{\approx} \tilde{\chi}^* TC$.

<u>Proof.</u>

Consider the following commutative diagram, with i,j inclusions and p,q,Π projections.

$$\begin{tikzcd}
C & X \times C \arrow[l, "q"'] \arrow[rr, "\tilde{F}"] \arrow[ddr, "p"] & & J^k(X) \arrow[ddl] \\
M \arrow[u, "\chi"] \arrow[ur, "i"] \arrow[rr, "\tilde{F}|M"] & & I^k(X) \arrow[ur, "j"] \arrow[d, "\Pi"] & \\
& & X &
\end{tikzcd}$$

The normal bundle to $I^k(X)$ in $J^k(X)$ is $\Pi^* T^* X$. Since $\tilde{F}$ is transverse to $I^k(X)$ and $M = \tilde{F}^{-1} I^k(X)$, the normal bundle to M in $X{\times}C$ is $\tilde{F}|^* \Pi^* T^*X$. Therefore, we have

$$TM \oplus \tilde{F}\,|^* \,\Pi^*\, T^*X \cong i^*\, T(X \times C)$$

and hence
$$\begin{aligned} TM \oplus \tilde{F}|^* \Pi^* T^* X &\cong i^*p^* TX \oplus i^*q^* TC \\ &\cong i^*p^* TX \oplus \chi^* TC. \end{aligned}$$

TX can be identified with T^*X, adding a complementary bundle to both sides we have $TM \overset{S}{\approx} \chi^* TC$. The second part is immediate.

□

<u>Corollary 1.</u>

$w_i(\tilde{M}) = \tilde{\chi}^*\, w_i(C)\ \forall\ i \geq 0.$

<u>Proof.</u>

The Stiefel Whitney classes are stable characteristic classes i.e. $w_i\,(E \oplus \varepsilon^k) = w_i(E)$ for any trivial bundle ε^k. So they are well defined

on stable equivalence classes of bundles. Therefore, $w_i(T\hat{M}) = w_i(\tilde{\chi}^* TC)$
$$= \tilde{\chi}^* w_i(TC).$$
$\square$

The above result is also true for other stable characteristic classes, such as integral Stiefel-Whitney classes or Pontrjagin classes.

Corollary 2 (Zeeman [33]).

C orientable $\Rightarrow$ $\hat{M}$ orientable.

Proof

The first Stiefel-Whitney class of a manifold is zero if and only if the manifold is orientable. So the required result is equivalent to $w_1(c) = 0 \Rightarrow w_1(\hat{M}) = 0$, but this follows from Corollary 1.

$\square$

Examples.

1 The complex projective plane $\mathbb{CP}^2$ does not occur as a component of a critical manifold if the parameter space is $\mathbb{R}^4$ since $w_2(\mathbb{R}^4) = 0$ and $w_2(\mathbb{CP}^2) \neq 0$.

2 Any non-orientable compact surface of odd genus can only occur as a component of a critical manifold when the parameter space is also a non-orientable compact surface of odd genus.

3 Milnor [19] proves that the total Stiefel-Whitney class of the r-dimensional projective space P^r is $(1+a)^{r+1}$ where a is the generator of $H^1(P^r)$. If the parameter space is $\mathbb{R}^r$ and $r+1$ is not a power of 2 then p^r cannot occur as a component of the critical manifold.

Definition.

For each connected manifold A the classifying map for its tangent bundle $\tau : A \to BO$ induces a map $\tau_* : \Pi_1(A) \to \Pi_1(BO) \cong Z_2$. The elements of $\Pi_1(A)$ lying in the Kernel of τ_* are called orientation preserving and the remainder are called orientation reversing.

Definition.

A map $g: A \to B$, with A,B connected manifolds is orientation true if $g_*: \Pi_1(A) \to \Pi_1(B)$ maps the orientation preserving and reversing elements of $\Pi_1(A)$ into the orientation preserving and reversing elements of $\Pi_1(B)$ respectively.

Let $\bar{\alpha}: S^1 \to A$ be an immersion representing an element α of $\Pi_1(A)$. The element α is orientation preserving if the normal bundle to $\bar{\alpha}(S^1)$ is orientable, otherwise, it is orientation reversing.

Theorem 8.

Let $\tilde{\chi} : \hat{M} \to C$ be a restricted catastrophe map, then $\tilde{\chi}$ is orientation true.

Proof.

Consider the following diagram

$$\begin{tikzcd} S^1 \arrow[r, "\bar{\alpha}"] & \hat{M} \arrow[d, "\tilde{\chi}"] \\ & C \end{tikzcd}$$

Let $\hat{N}$, N denote the normal bundles to S^1 in $\hat{M}$ and C respectively.

We have $TS^1 \oplus \hat{N} = \bar{\alpha}^* T\hat{M}$

and $TS^1 \oplus N = \bar{\alpha}^* \tilde{\chi}^* TC$.

Consider the first Stiefel-Whitney classes of these bundles

$$w_1(\hat{N}) = \bar{\alpha}^* \; w_1(\hat{M})$$

and $$w_1(N) = \bar{\alpha}^* \; \tilde{\chi}^* \; w_1(C).$$

By Corollary 1 $w_1(\hat{M}) = \tilde{\chi}^* \; w_1(C)$ and so $w_1(\hat{N}) = w_1(N)$. In particular $\hat{N}$ is orientable if and only if N is orientable.

An integer valued degree is defined for any proper map between manifolds of the same dimension (See Epstein [6]). However, when the map is orientation true we have the following definition due to Olum [20] .

<u>Definition</u>.

Let A and B be connected r-dimensional manifolds and $g: A \to B$ a proper map which is orientation true. Choose generators for $H_c^r(A, \tilde{Z})$ and $H_c^r(B, \tilde{Z})$, singular cohomology groups with twisted integer coefficients and compact supports. The induced map $g^*: H_c^r(B,\tilde{Z}) \to H_c^r(A,\tilde{Z})$ maps the generator of $H_c^r(B, \tilde{Z})$ onto an integer multiple of the generator of $H_c^r(A, \tilde{Z})$. We call this integer the <u>twisted degree</u> of g, which is written as deg g.

We can restrict to Z_2 coefficients and obtain the mod 2 degree in the same way, which we write as $\deg_2 g$. A degree can be assigned to the map $g: A \to B$ independent of the choices of generator for $H_c^r(A)$ and $H_c^r(B)$ by considering the absolute value of the degree.

From Olum [20] we obtain the following geometric interpretation of the twisted degree. Choose a regular value c of $g: A \to B$ such that there exists a disc D with $c \varepsilon D$ and the preimage of D is a finite number of disjoint discs $D_1, \ldots, D_k$ each mapped diffeomorphically by g onto D. Let $c_i \in D_i$ be the unique point such that $g(c_i) = c$. Choose an orientation at D_1 and D. For each disc D_i we define an integer δ^i which is +1 or -1 as follows: First, orient D_i so that g carries the positive orientation of D_i onto the positive orientation of D. Next, choose an arbitrary path ℓ from c_1 to c_i, and let $\delta_a^i = +1$ or -1 according as D_i is oriented coherently with D_1 along the path ℓ, or not ; let $\delta_b^i = +1$ or -1 according as the closed path into which ℓ is carried by g is orientation preserving or orientation reversing; then let $\delta^i = \delta_a^i \ \delta_b^i$. That δ^i is independent of the particular path ℓ used is a consequence of the fact that g is orientation true.

$$\deg \ g = \sum_{i=1}^{k} \delta^i$$

Olum [20] assumes his manifolds are compact, but $H_c^r(A) = \mathbb{Z}$ for any connected manifold A^r, so his reasoning carries over to the proper non-compact case.

In order to show how the degree of the a restricted catastrophe map influences the Euler characteristic of X, we need the following definition and two lemmas.

Definition.

Let $F: X^n \times C \to R$ be a good map and I the type in $J^k(X)$ induced by the Morse functions. A Morse sheet is a connected component of $\tilde{F}^{-1} I$. Associated to each Morse sheet is an index corresponding to one of the $n+1$ Morse orbits in $J^k(X)$.

finite number of disjoint discs $D_1, \ldots, D_k$ each mapped diffeomorphically by g onto D. Let $c_i \in D_i$ be the unique point such that $g(c_i) = c$. Choose an orientation at D_1 and D. For each disc D_i we define an integer δ^i which is +1 or -1 as follows: First, orient D_i so that g carries the positive orientation of D_i onto the positive orientation of D. Next, choose an arbitrary path ℓ from c_1 to c_i, and let $\delta^i_a = +1$ or -1 according as D_i is oriented coherently with D_1 along the path ℓ, or not ; let $\delta^i_b = +1$ or -1 according as the closed path into which ℓ is carried by g is orientation preserving or orientation reversing; then let $\delta^i = \delta^i_a \; \delta^i_b$. That δ^i is independent of the particular path ℓ used is a consequence of the fact that g is orientation true.

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<u>Definition</u>.

Let $F: X^n \times C \to R$ be a good map and I the type in $J^k(X)$ induced by the Morse functions. A <u>Morse sheet</u> is a connected component of $\tilde{F}^{-1} I$. Associated to each Morse sheet is an index corresponding to one of the n+1 Morse orbits in $J^k(X)$.

Lemma 8.

Let $F: X\times C \to \mathbb{R}$ be a good map type transverse to the type A_2 in $J^k(X)$. If F induces a restricted catastrophe map $\tilde{\chi} : \tilde{M} \to C$ with disjoint discs D_1 and D_i in $\tilde{M}$ mapped diffeomorphically onto a disc D in C by $\tilde{\chi}$, then

δ^i equals the difference between the Morse indices of D_1 and D_i (mod 2).

Proof.

The hypothesis that F is type transverse to A_2 implies that the singularities of $\tilde{\chi} : \tilde{M} \to C$ are fold catastrophes or singularities of codimension at least two. Therefore, we can choose a path from D_1 to D_i on $\tilde{M}$ which lies in Morse sheets or passes transversely through the fold catastrophes. Morse sheets are immersed in C by $\tilde{\chi}$, so as we traverse the path from D_1 to D_i along Morse sheets the relative orientation is preserved. Only when we pass through the fold singularities do we change the relative orientation. When we cross the fold curve the Morse index changes by ± 1, therefore, the number of times the path crosses the fold curve is equal to the difference between the Morse indices of D_1 and D_i (mod 2). The lemma follows since δ^i is the number of times we change the relative orientation.

$\square$

Lemma 9.

Let $F: X\times C \to \mathbb{R}$ be a good map, then there exists a good map $G: X\times C \to \mathbb{R}$ such that G is type transverse to the A_2 orbits in $I^k(X)$. Furthermore, M_F is diffeomorphic to M_G and for each of the corresponding connected components we have $\deg \tilde{\chi}_F = \deg \tilde{\chi}_G$.

Proof.

We homotopy F to G so that G is type transverse to the A_2 orbits in $I^k(X)$, whilst maintaining transversality to the Morse functions throughout the homotopy. Thus we have a diffeomorphism Φ from M_F to M_G and the following diagram commutes

$$\begin{tikzcd} M_F \arrow[rr, "\Phi"] \arrow[dr, "\chi_F"'] & & M_G \arrow[dl, "\chi_G"] \\ & C & \end{tikzcd}$$

Therefore for each connected component $\deg \tilde{\chi}_F = \deg \tilde{\chi}_G$.

□

Theorem 9.

Let X be compact and $F\colon X\times C^r \to \mathbb{R}$ be a good map with critical manifold $M^r = \coprod_{j=1}^{N} \tilde{M}_j$. We can choose generators for $H^r_c(C)$ and $H^r_c(\tilde{M}_j)$ so that the Euler characteristic of X is $\sum_{j=1}^{N} \deg \tilde{\chi}_j$.

Proof.

By Lemma 9 it suffices to consider a map type transverse to the A_2 orbits. Choose a regular value c of $\chi : M \to C$ such that there exists a disc D with $c\varepsilon D$ and the preimage of D is a finite number of disjoint discs $D_1,\ldots, D_k$ each mapped diffeomorphically by χ onto D. Choose an orientation at D. Let $c_i \varepsilon D_i$ be the unique point such that $\chi(c_i) = c$. Each D_i lies in a Morse sheet of index m_i say. Because c is a regular value, $F_c\colon X \to \mathbb{R}$ is a Morse function with Morse singularities at c_i of index m_i. This gives a cellular decomposition of X and hence the Euler characteristic of X is $\sum_{i=1}^{k} (-1)^{m_i}$.

For each restricted catastrophe map $\tilde{\chi}_j : \hat{M}_j \to C$ choose some c_i contained in $\hat{M}_j$, call it c_{ij}, c_{ij} lies in the disc D_{ij} with index m_{ij}. Choose an orientation at D_{ij} so that D_{ij} maps to $(-1)^{m_{ij}}$ times the orientation at D. We make this choice so that a disc lying in $\hat{M}_j$ with even index contributes $+1$ to the twisted degree of $\tilde{\chi}_j$, and a disc lying in $\hat{M}_j$ with odd index contributes -1. So that

$$\sum_{i=1}^{k} (-1)^{m_i} = \sum_{j=1}^{N} \deg \tilde{\chi}_j$$

and we have the required result.

□

<u>Example.</u>

Let S^2 and T^2 be the 2-sphere and the 2-torus respectively then any map $g: S^2 \to T^2$ has zero degree. So it could only occur as the catastrophe map for a compact state space X if the Euler characteristic of X is zero.

<u>Lemma 10.</u>

Let $F: X \times C^r \to \mathbb{R}$ be a good map with C compact, giving rise to a restricted catastrophe map $\tilde{\chi} : \hat{M} \to C$ with $\hat{M}$ compact. If $w_r(C) \neq 0$ then $w_r(\hat{M}) = \deg_2 \tilde{\chi}$.

<u>Proof.</u>

$\hat{M}$ and C are compact. By Corollary 1 of Theorem 7 $w_r(\hat{M}) = \tilde{\chi}^* w_r(C)$. The generator of $H_c^r(C, \mathbb{Z}_2)$ is $w_r(C)$ so its image in $H_c^r(\hat{M}, \mathbb{Z}_2)$ under $\tilde{\chi}^*$ is zero if and only if $w_r(\hat{M}) = 0$. So $\deg_2 \tilde{\chi} = w_r(\hat{M})$.

□

Example.

Let S^4 and $\mathbb{CP}^2$ denote the 4-sphere and the complex projective plane respectively. Since $w_4(\mathbb{CP}^2) \neq 0$ any restricted catastrophe map $\tilde{\chi} : S^4 \to \mathbb{CP}^2$ will have zero mod 2 degree.

So far we have not used the full information contained in Corollary 1 of Theorem 7, that for a restricted catastrophe map $w_i(\hat{M})=\tilde{\chi}^* \ w_i(C)$. To extract further information we can appeal to a geometric interpretation of the Stiefel-Whitney classes.

Definition.

Let Q be a manifold of dimension q. The p-th Stiefel-Whitney homology class of Q $w^p \epsilon H_p(Q)$ is the Poincare dual of $w_{q-p} \epsilon H^{q-p}(Q)$, the q-p th Stiefel-Whitney cohomology class.

Theorem 10 (Whitney).

Given a smooth triangulation of Q, take the barycentric subdivision and consider the infinite chain of p-simplices, these define a mod 2 cycle, the p-th Stiefel-Whitney homology class.

Proof.

See Halperin and Toledo [10] .

We can rewrite $w_i(\hat{M}^r)= \tilde{\chi}^* \ w_i(C^r)$ as $D\ w^{r-i}(\tilde{M})= \tilde{\chi}^* \ D\ w^{r-i}(C)$. Then Lemma 1 says that if $\tilde{\chi}$ is transverse to a cycle representing $w^{r-i}(C)$ then the preimage of this cycle under $\tilde{\chi}$ is a homology class representing $w^{r-i}(\hat{M})$.

Example.

The first Stiefel-Whitney homology class of the Klein bottle K^2

Example.

Let S^4 and $\mathbb{C}P^2$ denote the 4-sphere and the complex projective plane respectively. Since $w_4(\mathbb{C}P^2) \neq 0$ any restricted catastrophe map $\tilde{\chi} : S^4 \to \mathbb{C}P^2$ will have zero mod 2 degree.

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Example.

The first Stiefel-Whitney homology class of the Klein bottle K^2

is represented by a cycle going once round the core of both Möbius strips in K^2. Let $\tilde{\chi}: K^2 \to K^2$ be a restricted catastrophe map. When $\tilde{\chi}$ is transverse to any cycle in the parameter space K^2 going round both Möbius strips the preimage represents a cycle in the critical manifold K^2 also going round both Möbius strips.

§4 The extension of locally stable maps.

In this chapter we restrict attention to maps $F\colon X\times C^r \to \mathbb{R}$ with $r \leq 5$. By Thom's theorem (Trotman and Zeeman [30]) there exists an open dense set of maps such that $\hat{F}\colon X\times C \to J^k(X)$ is transverse to all the orbits in $J^k(X)$. The Scout's Honour Theorem (Markus [15]) asserts that these transversality conditions are equivalent to the local stability of $F\colon X\times C \to \mathbb{R}$ at all points $(x,c)\in X\times C$. We recall the definition of local stability.

Definition.

$F\colon X\times C \to \mathbb{R}$ is locally stable at (x,c) if $\forall$ nbds N of (x,c) in $X\times C$, $\exists$ a nbd $\mathcal{U}$ of F, such that $\forall$ $G\in\mathcal{U}$ $\exists\, (\tilde{x},\tilde{c})\in N$, so that $\exists$ diffeomorphsim germs α,β,γ making the following diagram commute.

$$\begin{tikzcd}
X\times C \arrow[r, "F\times\Pi"] \arrow[d, "\alpha"'] & \mathbb{R}\times C \arrow[r, "\Pi"] \arrow[d, "\beta"] & C \arrow[d, "\gamma"] \\
X\times C \arrow[r, "G\times\Pi"'] & \mathbb{R}\times C \arrow[r, "\Pi"'] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
(x,c) \arrow[d] \\
(\tilde{x},\tilde{c})
\end{tikzcd}$$

where Π denotes the appropiate projection map.

$F\colon X\times C \to \mathbb{R}$ is locally stable if it is locally stable at all points $(x,c)\in X\times C$.

The singularities of the catastrophe map associated to a locally stable map are elementary catastrophes [30]. As remarked at the end of Chapter one the Thom-Boardman types underlie the cuspoid elementary catastrophes so we have the following theorem.

Theorem 11.

Let $F: X \times C \to \mathbb{R}$ be a locally stable map inducing a restricted catastrophe map $\tilde{\chi}: \hat{M} \to C$. Then $\tilde{\chi}: \hat{M} \to C$ satisfies the hypothesis of the Thom polynomial theorem for the Thom-Boardman types Σ^{1^q} in $J^k(\hat{M},C)$.

Proof.

Since F is locally stable, $\hat{F}$ is transverse to the orbits $\pm A^i_j$ in $J^k(X)$. The singularities of $\tilde{\chi}$ are determined by $\hat{F}$. If $\tilde{\chi}$ drops rank 1 at m then $\hat{F}(m) \varepsilon \pm A^i_j$ for some i and j. Let $m \varepsilon \tilde{F}^{-1}(\pm A^i_j)$, by Mather's theory of unfoldings (Trotman and Zeeman [30] Chapter 7), the equivalence class of the germ of $\tilde{\chi}$ at m, $\tilde{\chi}_m$ is determined by $\pm A^i_j$. Mather's theory also enables us to compute the germ $\tilde{\chi}_m$ and this is a stable Thom-Boardman singularity of type $\Sigma^{1^{(j-1)}}$ (See Golubitsky and Guillemin [7] page 177). Therefore $\tilde{\chi}^k$ is transverse to Σ^{1^q} in $J^k(\hat{M},C)$. $F: X \times C \to \mathbb{R}$ is locally stable and so F is transverse to all orbits in $J^k(M,C)$. That is, $\tilde{\chi}$ satisfies the hypothesis of Theorem 1, Corollary 1 for the types Σ^{1^q}.

□

We now show how our methods shed light on some of the extension problems discussed by Markus [15].

Let P^r be a compact connected manifold with boundary ∂P embedded in a manifold of the same dimension Q^r, so that it has an open neighbourhood $\hat{P}$ in Q^r which is a manifold without boundary. Given a locally stable map $F: X \times \partial P \to \mathbb{R}$, Markus [15] proves that there exists a locally stable extension $G: X \times \hat{P} \to \mathbb{R}$.

$$\text{i.e.} \qquad G\Big|_{X \times \partial P} = F \; .$$

Following Markus we pose the question : Do there exist extensions with no isolated singularities? As Markus has pointed out it is sensible to impose some compactness condition. In fact, to apply our results we require a compactness restriction on the extension, since we want to use the zero'th homology group to count singularities.

<u>Theorem 12.</u>

Let P^r and Q^r be as above and suppose that we have two locally stable extensions

$$G_1, G_2 : X \times P \to \mathbb{R}$$

with M_{G_1} and M_{G_2} compact sets above P.

Let Σ be a type in $I^k(X)$ of codimension r. Then the mod 2 number of singularities of type Σ above P is the same for G_1 and G_2 .

<u>Proof.</u>

We first double the manifold P to obtain a closed manifold $2P$. We then define a locally stable map $G\colon X \times 2P \to \mathbb{R}$ by the maps G_1 and G_2 on each half. By hypothesis M_{G_1} and M_{G_2} are compact, so M_G is compact. Theorem 4 shows that for a type Σ of codimension r the number of singularities of type Σ of G on each component of M_G is even. That is, the number of singularities of type Σ for G_1 is equal mod 2 to the number of singularities of type Σ for G_2.

$\square$

<u>Example.</u>

Let $F\colon X \times S^2 \to \mathbb{R}$ be locally stable with X compact and S^2 the two dimensional sphere. Consider S^2 embedded in $\mathbb{R}^3$. Extend F to a locally stable function parametrised by an open neighbourhood of the closed 3-ball. The mod 2 number of each of the isolated singularity types, i.e. umbilics of each index and swallowtails, is dependent on F only and is independent of the particular extension.

The above theorem as in the extension problems discussed by Markus depend on the locally stable map parametrised by ∂P. The work of Zeeman [33] is of an essentially different kind. In fact the extension problem dealt with by Zeeman depended on the critical manifold only and not the locally stable map or the state space. However, to extend the results of Zeeman we need to put more restrictions on P. The results only apply to the isolated cuspoids and we require P to be orientable and the dimension of P to be less than five. We first prove two lemmas.

<u>Lemma 11.</u>

Let A_1^r and A_2^r be compact orientable r-manifolds with boundary, together with a diffeomorphism $\partial A_1 \cong \partial A_2$. Let A be the manifold obtained by glueing A_1 to A_2 with the diffeomorphism. Then $w_1(A)^2 = 0$, where $w_1(A)$ is the first Stiefel-Whitney class of A.

<u>Proof.</u>

Choose an orientation on A_1 and A_2. By Theorem 10 we can compute the Stiefel-Whitney class dual to $w_1(A)$. This homology class is equal to the sum of the r-1 dimensional cycles representing each component of ∂A_1 where the diffeomorphism does not preserve orientation. Using a collar on A_1 we can slide the homology class off of ∂A_1 and hence the cup product of $w_1(A)$ with itself is zero.

$\square$

<u>Lemma 12.</u>

Let $f\colon A^r \longrightarrow B^r$ $1 \leqslant r \leqslant 4$ be a map between compact connected manifolds A and B such that $w_1(A)^2 = 0$ and $w_1(B) = 0$. If f is type transverse to the Thom-Boardman type Σ^{1^r} in $J^r(X,Y)$ then the number of isolated singularities of type Σ^{1^r} is even.

Proof.

We consider each r and show that the required Thom polynomial is zero. For the cases $1 \leq r \leq 3$ the Thom polynomials have been computed by Porteous [22], however to prove this lemma we show that all polynomials of codimension r in the Stiefel-Whitney classes of TA and f^*TB are zero.

r=1 S^1 is orientable so $w_1(S^1) = 0$.

r=2 For any compact 2-manifold Y the total Stiefel-Whitney class is $W(Y) = 1 + w_1(Y) + w_1(Y)^2$, so $W(A) = 1 + w_1(A)$, $W(B) = 1$.
So any polynomial of codimension 2 is zero.

r=3 Every compact 3-manifold Y immerses in R^4 so the total Stiefel-Whitney class is $W(Y) = 1 + w_1(Y) + w_1(Y)^2$, so $W(A) = 1 + w_1(A)$, $W(B) = 1$.
Therefore any polynomial of codimension 3 is zero.

r=4 The Thom polynomial for $\Sigma^{1,1,1,1}$ was computed by Sergeraert [26] using an algorithm due to Ronga. The polynomial is $c_1(c_1^3 + c_3)$ where c_i are the Stiefel-Whitney classes of the difference bundle $f^*TB - TA$.
For any compact orientable 4-manifold B we have

$$W(B) = 1 + w_2(B) + w_2(B)^2 \quad \text{(See Massey [16]).}$$

$$\begin{aligned} W(f^*TB-TA) &= (1 + f^*w_2(B) + f^*w_2(B)^2)(1 + \bar{w}_1(A) + \bar{w}_2(A) + \bar{w}_3(A)) \\ &= (1 + f^*w_2(B) + f^*w_2(B)^2)(1 + w_1(A) + w_2(A) + w_3(A)), \end{aligned}$$

since $w_1(A)^2 = 0$.

$$c_1(c_1^3 + c_3) = w_1(A)(w_1(A)^3 + w_1(A)f^*w_2(B) + w_3(A))$$
$$= w_1(A)\, w_3(A)$$

The generators of the 4th unoriented cobordism group have Stiefel-Whitney number $w_1 w_3 = 0$ (Milnor [19]), that is, the product of the first and third Stiefel-Whitney class of any 4-manifold is zero. In particular $w_1(A)\, w_3(A) = 0$.

□

We note that the corresponding result to Lemma 11 when $r=5$ is false. As above we assume P^r is a compact connected manifold with boundary ∂P embedded in a manifold of the same dimension Q^r, so that P has an open neighbourhood $\hat{P}$ in Q^r which is a manifold without boundary. We further assume that P is orientable and that $1 \leq r \leq 4$.

<u>Theorem 13.</u>

Let G be any locally stable map parametrised by $\hat{P}$ whose critical manifold M_G above the orientable manifold P^r is compact and G is the extension of a locally stable map parametrised by ∂P whose critical manifold is M. Let N be an orientable compact manifold with boundary ∂N. Suppose there exists a map $f\colon N^r \to P^r$ such that:

(i) There exists a diffeomorphism Φ making this diagram commute

$$\begin{tikzcd}
\partial N \arrow[rr, "\Phi"] \arrow[dr, "f|_{\partial N}"'] & & M \arrow[dl, "\mathrm{proj}"] \\
& \partial P &
\end{tikzcd}$$

(ii) The germ of f at all $n \in N - \partial N$ is stable.

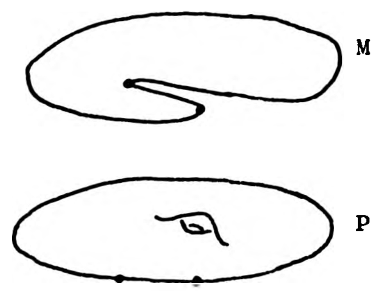
Then the mod 2 number of singularities of type Σ^{1r} above P is the same for f and the projection of M_G.

Proof.

We double the manifold P to obtain a closed orientable manifold $2P$. We also glue N and M_G along their boundary M to obtain a closed manifold M^*. M^* is not necessarily connected, but it will have a finite number of connected components $\{M_i^*\}_{i=1}^n$. We consider the maps $p_i : M_i^* \to 2P$ defined by the restriction of the projection $M_G \to P$ and the map $N \to P$. The maps p_i have generic singularities of type Σ^{1^r} by Theorem 11 and the hypothesis on N. By construction the maps p_i and the manifolds M_i^* satisfy the hypothesis of Lemma 12. So we have an even number of singularities of type Σ^{1^r} on each M_i^* and hence an even number on M^*.

Example.

Let P be the torus with an open disc removed so that ∂P is a circle. Suppose that we have the following picture with the critical manifold M above ∂P also a circle, but projecting onto ∂P with two fold catastrophes as shown.



It is easy to fill in M with a compact orientable manifold N mapping into P with one cusp. So by the theorem, the catastrophe map of any locally stable map extending over P will contain an odd number of cusp catastrophes, provided that the new critical manifold above P is compact.

§5 Applications to Lagrangian immersions.

We now study the singularities of real valued functions defined on fibre bundles. We also discuss the connection with Lagrangian immersions.

Let X^n and C^r be connected manifolds of dimension n and r respectively. We denote by $X \tilde{\times} C$ the total space of a fibre bundle with fibre X and base space C. Let $\{(U_i, \varphi_i)\}$ be a trivialising cover for the bundle $q: X \tilde{\times} C \longrightarrow C$. That is $\{U_i\}$ is an open cover of C and $\varphi_i : q^{-1}U_i \dashrightarrow U_i \times X$ are diffeomorphisms.

$$\begin{tikzcd}[row sep=small] q^{-1}U_i \arrow[rr, "\varphi_i"] & & U_i \times X \\ y \arrow[rr, mapsto] \arrow[dr] & & q(y), \hat{\varphi}_i(y) \arrow[dl] \\ & U_i & \end{tikzcd}$$

Let $\{(V_j, \psi_j)\}$ be a maximal atlas for the fixed copy of X then each $\psi_j : V_j \longrightarrow \psi_j(V_j) \subset \mathbb{R}^n$ is a homeomorphism. We construct a jet bundle over $X \tilde{\times} C$ with fibre J^k the jets of order k $\mathbb{R}^n, 0 \to \mathbb{R}, 0$

$$J^k = L(\mathbb{R}^n, \mathbb{R}) \times L(\circ^2\mathbb{R}^n, \mathbb{R}) \times \ldots\ldots \times L(\circ^k\mathbb{R}^n, \mathbb{R})$$

and group action $\mathcal{G}_n^k$, the group of invertible jets of order k $\mathbb{R}^n, 0 \to \mathbb{R}^n, 0$.

Let s be a section of the trivial line bundle over $X \tilde{\times} C$. Let $y \in X \tilde{\times} C$ with $y \in q^{-1}U_i$ and $\hat{\varphi}_i(y) \in V_j$ then we have the following diagram.

$$\begin{array}{ccccccc} q^{-1}U_i & \longleftrightarrow & U_i \times X & \longleftarrow & U_i \times V_j & \longrightarrow & U_i \times \psi_j(V_j) \\ y & \longleftarrow\!\!\shortmid & q(y), \hat{\varphi}_i(y) & & \longmapsto & & q(y), \psi_j \hat{\varphi}_i(y) \end{array}$$

The section s induces a map $s_{ij} : q(y) \times \psi_j(V_j) \longrightarrow \mathbb{R}$.

We now define an equivalence relation for each positive integer k. Let s and t be sections and y_1 and y_2 points of $X \tilde{\times} C$ such that there exists an i and j with $y_1, y_2 \in q^{-1}U_i$ and $\hat{\varphi}_i(y_1), \hat{\varphi}_i(y_2) \in V_j$.

$(s,y_1) \overset{k}{\sim} (t,y_2) \iff y_1 = y_2$ and the k-jets of the induced maps s_{ij} and t_{ij} are equivalent at $\psi_j\hat{\varphi}_i(y_1)$.

Let $J^k\tilde{\times} C$ denote the set of all equivalence classes (s,y). This is well defined and has a natural vector bundle structure with fibre J^k and group action $\mathcal{G}^k_n$.

A map $F:X\tilde{\times} C\to\mathbb{R}$ has a natural k-jet extension

$$\tilde{F}^k : X\tilde{\times} C \longrightarrow J^k\tilde{\times} C$$
$$y \longmapsto [F,y]$$

In this more general situation we can define as before, good maps, the critical manifold, the catastrophe map, type transversality, etc..

The jet bundle $J^1\tilde{\times} C$ can be identified with the dual of the tangent bundle along the fibres of $q:X\tilde{\times} C\to C$ and hence we have the relationship,

$$J^1\tilde{\times} C \oplus q^*TC = T(X\tilde{\times} C).$$

We obtain the following results corresponding to Theorem 3 and Theorem 7.

<u>Theorem 14</u>.

Let $F:X\tilde{\times} C\to\mathbb{R}$ be a good map type transverse to a type Σ in $J^k\tilde{\times} C$. Then the closure of the set of singularities of type Σ carries a fundamental class whose Poincaré dual after inclusion in the homology of $X\tilde{\times} C$ is given by a polynomial in the Stiefel-Whitney classes of $J^1\tilde{\times} C$.

<u>Proof</u>.

Consider the following commutative diagram where h is a classifying map for $J^k\tilde{\times} C \dashrightarrow X\tilde{\times} C$ into a Grassmannian G_n.

$$\begin{tikzcd} J^k \tilde{\times} C \arrow[r, dash] & E \arrow[d] \\ X \tilde{\times} C \arrow[u, dashed, "\tilde{F}^k"] \arrow[r, dashed, "h"] & G_n \end{tikzcd}$$

we have $J^k \tilde{\times} C = h^* E$. A type Σ of $J^k \tilde{\times} C$ induces a type in E. The cohomology class representing the closure of the set of singularities of type Σ in $X \tilde{\times} C$ arises as the pullback of a cohomology class in E, using the commutative diagram we see that the pullback of this class factors through G_n. The map h is also the classifying map for $J^1 \tilde{\times} C$. Therefore the induced map on cohomology pulls back the generators of $H^*(G_n)$ to the Stiefel-Whitney classes of $J^1 \tilde{\times} C$.

□

<u>Theorem 15</u>.

If $F: X \tilde{\times} C \to R$ is a good map then $TM \overset{s}{\sim} \chi^* TC$ and $T\breve{M} \overset{s}{\sim} \chi^* TC$.

<u>Proof</u>.

Consider the following commutative diagram,

$$\begin{tikzcd} M \arrow[r, "i"] \arrow[dr, "\chi"'] & X \tilde{\times} C \arrow[r, dash, "\tilde{F}"] \arrow[d, "q"'] & J^1 \tilde{\times} C \\ & C & \end{tikzcd}$$

The normal bundle to M in $X \tilde{\times} C$ is $i^*(J^1 \tilde{\times} C)$, since $\tilde{F}$ is transverse to the zero section of $J^1 \tilde{\times} C$.

Therefore,

$$TM \oplus i^*(J^1 \tilde{\times} C) = i^* T(X \tilde{\times} C)$$
$$TM \oplus i^*(J^1 \tilde{\times} C) = i^*(J^1 \tilde{\times} C) \oplus \chi^* TC.$$

So we have the required result $TM \overset{s}{\sim} \chi^* TC$ and this also applies to each connected component.

□

The corollaries to Theorem 7 also apply to a good map $F: X\tilde{\times}C \to \mathbb{R}$ so we have in particular that $w_i(\hat{M}) = \tilde{\chi}^* w_i(C) \quad i \geqslant 0$.

<u>Definition</u>.

Let L and C be r-dimensional connected manifolds. We recall that there exists a canonical 2-form, the symplectic form, on the cotangent bundle T^*C and that an immersion $j: L^r \to T^*C^r$ is called <u>Lagrangian</u> if the pullback of this 2-form is zero. Given an immersion $j: L \to T^*C$ there exists a map $\chi : L \to C$ induced by composing j with the natural projection $p: T^*C \to C$.

$$\begin{tikzcd} L \arrow[r, "j"] \arrow[dr, "\chi"'] & T^*C \arrow[d, "p"] \\ & C \end{tikzcd}$$

Weinstein ([31] lecture 6) proves that a good map $F: X\tilde{\times}C \to \mathbb{R}$ (Weinstein uses the term Morse family for a good map) induces a Lagrangian immersion of each component of the critical manifold M into T^*C and also that locally the converse is true. That is, if L is a Lagrangian manifold in T^*C, at each point $x \in L$ there exists an open neghbourhood which coincides with an open neighbourhood of a critical manifold for some good map parametrised by C.

Globally the converse is false. A cohomology class in $H^1(L;\mathbb{Z})$ can be defined for a Lagrangian immersion $L \to T^*C$, called the Maslov class (see Weinstein [31] page 27), which is zero if L is the critical manifold of a good map. This class is non-zero for an embedded circle in $T^*\mathbb{R}$, (See Arnold [1]), so an embedded circle in $T^*\mathbb{R}$ cannot arise from a good map. The following relation is satisfied by arbitrary Lagrangian immersions.

Theorem 16.

If $j: L \to T^*C$ is a Lagrangian immersion then $w_i^2(L) = \chi^* w_i^2(C) \quad i \geq 0$.

Proof.

Weinstein ([31] page 7) shows that if $j: L \to T^*C$ is a Lagrangian immersion then the Lagrangian plane $T_x L$ is complemented by a Lagrangian plane in $j^* T_{j(x)} T^* C$, which can be identified with $T_x L^*$ using the symplectic structure. So we have

$$TL \oplus T^*L \cong j^* T\, T^* C\ .$$

We can rewrite this as

$$\begin{aligned} TL \oplus T^*L &\cong j^*(p^* T^* C \oplus p^* TC) \\ &\cong \chi^* T^* C \oplus \chi^* TC. \end{aligned}$$

The total Stiefel-Whitney class then gives

$$\begin{aligned} W(L).\ W(L) &\cong \chi^* W(C).\ \chi^* W(C) \\ W(L)^2 &\cong \chi^* W(C)^2 \end{aligned}$$

So $w_i^2(L) = \chi^* w_i^2(C)$.

□

This result should be contrasted with the remark after Theorem 15 that $w_i(\tilde{M}) = \tilde{\chi}^* w_i(C)$ for a component of the critical manifold arising from a good map $F: X \tilde{\times} C \to \mathbb{R}$.

Example.

If L^2 immerses as a Lagrangian manifold in $T^*\mathbb{R}^3$ then $w_1^2(L^2) = w_2(L^2) = 0$, so L has even Euler characteristic if it is compact. Therefore the projective plane does not immerse as a Lagrangian manifold in $T^*\mathbb{R}^2$.

In the Appendix an explicit Lagrangian immersion of the Klein bottle K^2 in $T^*\mathbb{R}^2$ is given , since K^2 is non-orientable Corollary 2 of Theorem 7 says that it cannot occur as a critical manifold when the parameter space is $\mathbb{R}^2$.

We remark that all the results on catastrophe maps give necessary conditions for a given Lagrangian immersion to arise from a good map.

§6 Thom polynomials in cobordism theory.

The cohomology theory derived from the Thom spectrum MO has the following geometric interpretation [24] , [12] .

For a manifold X define the unoriented cobordism group $N^k(X)$ $k \in \mathbb{Z}$ as follows:

Consider the pairs (M,f) with M a manifold such that $\dim M = \dim X - k$ and $f: M \to X$ is a proper map. We define an equivalence relation between such pairs as follows:

$(M,f) \sim (N,g)$ if there exists a manifold W with boundary ∂W and a proper map $h: W \to X$ such that ∂W is diffeomorphic to $M \amalg N$, where $\amalg$ is the disjoint union, and

$$h|_M = f \ , \quad h|_N = g \ .$$

We define addition in $N^k(X)$ by

$$[(M,f)] + [(N,g)] = [(M \amalg N,\ f \amalg g)]$$

With this binary operation $N^k(X)$ becomes an abelian group (defined to be 0 if $k > \dim X$) in which every element has order 2.

$N^*(-)$ is a homotopy functor on the category of manifolds and maps; given $g: Y \to X$ we define

$$g^* : N^*(X) \to N^*(Y) \qquad \text{as follows,}$$

Let $[(M,f)] \in N^k(X)$. Choose a proper homotopy of f, say $\tilde{f}$, so that $\tilde{f}$ is transverse to g. We have the following pullback square.

$$\begin{tikzcd}
M \times_X Y \arrow[r, "q"] \arrow[d] & Y \arrow[d, "g"] \\
M \arrow[r, "\tilde{f}"] & X
\end{tikzcd}$$

Define $g^*[(M,f)] = [M \times_X Y, q] \in N^k(Y)$, where q is the natural map of the fibre product into Y.

A product $\cup : N^p(X) \times N^q(X) \to N^{p+q}(X)$ is defined by the composition

$$N^p(X) \times N^q(X) \xrightarrow{\times} N^{p+q}(X \times X) \xrightarrow{\Delta^*} N^{p+q}(X)$$

where

$$[M,f] \times [N,g] = [M \times N, f \times g]$$

and

$\Delta : X \to X \times X$ is the diagonal map.

Let $f: Y \to X$ be a proper map with $\dim X - \dim Y = k$, then the <u>umkehr</u> of f denoted $f!$ is the homomorphism

$$f! : N^q(Y) \to N^{q+k}(X)$$

$$[N,g] \to [N, f \circ g]$$

$f!$ is a group homomorphism $f!(f^* x \cup y) = x \cup f!(y)$ whenever $x \in N^*(X)$ and $y \in N^*(Y)$.

If f,g are proper and the following diagram commutes then

$$\begin{array}{ccc} Z \times_X Y & \xrightarrow{\bar{g}} & Y \\ \downarrow{\scriptstyle \bar{f}} & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow[g]{} & X \end{array} \qquad\qquad f^* \circ g! = \bar{g}! \circ \bar{f}^*$$

(see [24]).

Let $E \to X$ be a vector bundle with $z: X \to E$ the zero section. The Euler class of E is defined to be $e(E) = z^* z! [X, id]$.

Generalising the Stiefel-Whitney characteristic classes for Z_2 cohomology we have the following characterising properties of the Conner-Floyd characteristic classes.

To each n-plane bundle $E \to X$ there corresponds a sequence of cobordism classes.

$$w_i(E) \varepsilon N^i(X) \qquad i = 0,1,2,\ldots, n$$

$$w_0(E) = 1 \quad \text{such that}$$

(a) For a map $g: Y \to X$ $\qquad w_i(g^*E) = g^* w_i(E)$

(b) $w_k(E \oplus F) = \sum_{i+j=k} w_i(E).\ w_j(F)$

(c) If L is a line bundle $w_1(L) = e(L)$.

These characteristic classes arise as the pullbacks of generators of the cobordism ring of the classifying space for the bundle. Calculations involving cobordism characteristic classes are complicated because the Euler class of a tensor product of two line bundles is given by a formal group law (see [12]). That is, if L,M are line bundles over X.

$$e(L \otimes M) = \sum_{i,j} \eta_{ij}.\ e(L)^i.\ e(M)^j$$

$\eta_{ij} \varepsilon N^{1-i-j}(X)$ are represented by manifolds with constant maps. We can now define tom Dieck characteristic classes which contain information about the tensor product.

To each n-plane bundle $E \to X$ there corresponds a sequence of cobordism classes $\hat{w}_i(E) \varepsilon N^i(X)$ $i=n,n-1,\ldots$

Let $\hat{W}(E) = \sum_{i<n} \hat{w}_i(E)$ be the total tom Dieck class then

(a) For a map $g: Y \to X$ $\qquad \hat{W}(g^*E) = g^* \hat{W}(E)$.

(b) $\hat{W}(E \oplus F) = \hat{W}(E)\hat{W}(F)$.

(c) If L is a line bundle $\hat{W}(L) = \sum_{i,j} \eta_{ij}\, e(L)^i$.

We can now define a Steenrod-tom Dieck operation for each integer q.

$$R^q: N^k(Y) \to N^{k+q}(Y)$$
$$[N,g] \to g!\,\hat{w}_q(g^*TY-TN)$$

The R^q have the following properties.

(a) $R^q(x) = x^2$ if $x \in N^q(Y)$.

(b) $R^q(x) = 0$ if $x \in N^p(Y)$ and $p < q$.

(c) $R^q(x\cdot y) = \sum_{i+j=q} R^i(x).\ R^j(y)$.

We also have a Riemann-Roch theorem, for a proper map $f: X \to Y$,

$$R^q f!\ [M,g] = f! \sum_{i+j=q} \hat{w}_i(f^*TY-TX).\ R^j\ [M,g] \ .$$

The relation between unoriented cobordism and $\mathbb{Z}_2$ cohomology is given by the following isomorphism (see [24]).

$$N^*(X) \cong H^*(X) \otimes_{\mathbb{Z}_2} N^*(pt)$$

$N^*(pt)$ is a polynomial ring over $\mathbb{Z}_2$ with a generator of degree $-i$ for each positive integer not of the form 2^k-1. The isomorphism induces an epimorphism $\mu : N^*(X) \to H^*(X)$. Under this epimorphism both the Conner-Floyd and the tom Dieck characteristic classes map to the Stiefel-Whitney characteristic classes.

Also $Sq^q \circ \mu = \mu \circ R^q$ (See [29] (15.2)) gives the relation of the R^q to the Steenrod squares. So we can give the Steenrod squares the following geometric interpretation.

$$Sq^q: H^k(Y) \to H^{k+q}(Y)$$
$$\mu[N,g] \to g!w_q(g^*TY-TN)$$

where [N,g] is a cobordism class representing the element of $H^k(Y)$. On the righthand side we have $g!$ the umkehr map on cohomology and w_q is the q-th Stiefel Whitney class.

The Steenrod squares also satisfy the Riemann-Roch theorem as for the Steenrod-tom Dieck operations.

We remark that because of the isomorphism given above connecting cobordism with $\mathbb{Z}_2$ cohomology we have the following result. Let G_n be a Grassmannian of n-planes in $\mathbb{R}^N$. N can be chosen large enough, so that there are no relations between the generators of N^*G_n in a fixed codimension r.

We now prove a generalisation of the Thom polynomial theorem for cobordism theory in the case where the singularity set of a given type is a closed submanifold. By a closed submanifold we mean a manifold which is embedded as a closed set. We need to restrict ourselves to types which can be resolved as follows.

<u>Definition.</u>

Let Σ be a type in $J^k(X,Y)$. Consider the following commutative diagram, where h is a classifying map for $J^k(X,Y)$.

$$\begin{tikzcd} J^k(X,Y) \arrow[r] \arrow[d] & E \arrow[d] \\ X \times Y \arrow[r, "h"] & G_n \times G_m \end{tikzcd}$$

We choose $G_n \times G_m$ as a classifying space, where G_n and G_m are the Grassmannians of n and m planes in some $\mathbb{R}^N$ for N large. $J^k(X,Y) = h^*E$. The bundle E has the same fibre and group action as $J^k(X,Y)$ and so we have a type Σ^* in E such that $\Sigma = h^* \Sigma^*$.

We define a resolution of Σ to be a proper map $p\colon \hat{\Sigma} \to \overline{\Sigma}^*$ from a manifold $\hat{\Sigma}$ without boundary onto the closure of Σ^* which is a diffeomorphism restricted to $p^{-1}\Sigma^*$.

We conjecture that there exists a resolution for every type Σ.

Theorem 17.

Let Σ be a type in $J^k(X,Y)$ with a resolution $p\colon \hat{\Sigma} \to \overline{\Sigma}^*$. If $f\colon X \to Y$ is a map type transverse to Σ and $f^{k^{-1}}\Sigma$ is a closed submanifold, then the cobordism class $[\Sigma(f), i] \in N^*X$, where i denotes the inclusion is given by a universal formula in the Conner-Floyd characteristic classes of TX and f^*TY.

Proof.

Consider the following two commutative diagrams.

$$\begin{tikzcd}
 & & & & \hat{\Sigma} \arrow[d, "p"'] \\
 & & J^k(X,Y) \arrow[r, "\psi"] \arrow[d, "\Pi"'] & E \arrow[d, "\phi"] & \overline{\Sigma}^* \arrow[l, hook', "j"'] \\
\Sigma(f) \arrow[r, hook, "i"] & X \arrow[ur, "f^k"] \arrow[r, "f^{\circ}"] \arrow[dr, "f"'] & X \times Y \arrow[r, "h"] \arrow[d, "p_2"] & G_n \times G_m & \\
 & & Y & &
\end{tikzcd}$$

Where h is the classifying map as above, i and j are inclusions, Π and ϕ are bundle projections and p_2 projection onto the second factor.

$$\begin{tikzcd}
X \arrow[r, "\pi_1"] & G_n \\
X \times Y \arrow[u, "p_1"] \arrow[r, "h"] \arrow[d, "p_2"'] & G_n \times G_m \arrow[u, "\overline{p}_1"'] \arrow[d, "\overline{p}_2"] \\
Y \arrow[r, "\pi_2"] & G_M
\end{tikzcd}$$

Let $p_1, p_2, \overline{p_1}$ and $\overline{p_2}$ be projections onto factors of a product. We denote by π_1 and π_2 classifying maps for ξ^k_n and ξ^k_m respectively, i.e. classifying maps for the tangent bundles of X and Y.

Let r be the codimension of Σ in $J^k(X,Y)$. We first prove that if $\gamma \in N^r(G_n \times G_m)$ then $h^*\gamma$ is a unique polynomial in $p_1^* w_i(X)$ and $p_2^* w_j(Y)$. The groups $N^*(G_n)$ and $N^*(G_m)$ are generated by the Conner-Floyd classes w_i' and w_j' of their respective cannonical bundles. The Grassmannians were chosen so that there were no relations between these classes in codimension r. Therefore by the Künneth theorem there exists a unique polynomial P such that

$$\gamma = P(\overline{p}_1^* w_i', \overline{p}_2^* w_j'), \text{ from which we have}$$

$$h^* \gamma = P(h^* \overline{p}_1^* w_i', h^* \overline{p}_2^* w_j')$$

$$= P(p_1^* w_i(X), p_2^* w_j(Y)).$$

Now that this is established we return to the first diagram. The map ψ is a bundle map and so is transverse to $\overline{\Sigma}^*$. Therefore, we have the fibre product and induced map $q\colon J^k(X \times Y) \times_E \hat{\Sigma} \to J^k(X \times Y)$.

Where q is a diffeomorphism restricted to $q^{-1}\Sigma$.

By hypothesis f^k is transverse to Σ with $f^{k^{-1}}\,\overline{\Sigma}$ a manifold.

Therefore, we have the fibre product and induced projection $\overline{q}$, which factors through $\Sigma(f)$ and the fibre product is diffeomorphic to $\Sigma(f)$.

$$\begin{tikzcd}
X \times_J (J^k(X \times Y) \times_E \hat{\Sigma}) \arrow[d] \arrow[dr, "\overline{q}"] & \\
\Sigma(f) \arrow[r, "i"] & X
\end{tikzcd}$$

We now consider $[\Sigma(f), i] \varepsilon N^r(X)$, that is, the manifold of singularities of type Σ as an element of the cobordism group $N^r(X)$.

$$[\Sigma(f), i] = f^{k*}[J^k(X \times Y)\times_E \hat{\Sigma}, q]$$
$$= f^{k*} \psi^*[\hat{\Sigma} , j \circ p]$$

The fibre of E is contractible, hence ϕ^* is an isomorphism. It follows that there exists a unique class $\gamma \varepsilon N^r(G_n \times G_m)$ such that $\phi^*(\gamma) = [\hat{\Sigma} , j \circ p]$.

Therefore, by the commutativity of the first diagram

$$[\Sigma(f), i] = f^{k*} \psi^* \phi^{*-1} \gamma$$
$$= f^{\circ *} h^* \gamma .$$

By the first part we know that $h^*\gamma$ can be expressed as a unique polynomial $P(p_1^* w_i(X), p_2^* w_j(Y))$.

Hence,

$$[\Sigma(f), i] = f^{\circ *} P(p_1^* w_i(x), p_2^* w_j(Y))$$
$$= P(w_i(X), f^* w_j(Y)).$$

□

The proof of Theorem 17 was complicated because we worked with finite Grassmannians. This complication was introduced so that the resolution problem for a given type Σ is more accessible. For example, if in each fibre the closure of Σ is a real algebraic set then the closure of the type Σ in E is real algebraic and in particular is real analytic. Of course particular resolutions have been given for the Thom-Boardman types Σ^1 and $\Sigma^{1,1}$ by Ronga [25] , these resolutions are stronger than required in the theorem since they respect the stratification of the closure. In Chapter seven we consider the resolution of the Σ^1 .

The Jet bundle $J^k(X,Y)$ for all manifolds X^n, Y^m of dimension n,m has the same fibre and therefore the same resolved type above $G_n \times G_m$ independent of X and Y. The formula is universal in the sense that it is independent of the particular manifolds X and Y.

A formula involving the Conner-Floyd characteristic classes of $w_i(X)$ and $f^* w_j(Y)$ consists of a polynomial in $w_i(X)$ and $f^* w_j(Y)$ for every generator of $N^*(pt)$. We call the formula a cobordism Thom polynomial.

Corollary 1.

The epimorphism $\mu : N^*(X) \to H^*(X)$ maps the cobordism Thom polynomial for a type Σ with a resolution onto the cohomology Thom polynomial for the type Σ.

□

Corresponding to Theorem 1 we have the following corollary.

Corollary 2.

Let $F\colon X \times C \to Y$ be a parametrised family of maps type transverse to a type Σ in $J^k(X,Y)$, with a resolution. If the set of singularities of type Σ is a closed submanifold then the cobordism class defined by the inclusion of this manifold in $X \times C$ is given by a universal formula in $F^* w_i(X)$ and $p^* w_j(Y)$, where $w_i(X)$ and $w_j(Y)$ are the Conner-Floyd characteristic classes of X and Y and $p\colon X \times C \to X$ is the projection.

Proof.

This follows immediately from the theorem and the commutative diagram

$$\begin{tikzcd} & J^k(X,Y) \arrow[d] \\ X \times C \arrow[ur, "\tilde{F}^k"] \arrow[r, "\tilde{F}^\circ"'] & X \times Y \end{tikzcd}$$

□

In Chapter eight we consider parametrised real valued functions.

<u>Corollary 3.</u>

Let X be compact and $f: X \to Y$ a map satisfying the hypothesis of Theorem 17 for a type Σ in $J^k(X,Y)$. Then as an abstract manifold, the singularities of type Σ, can be considered as an element of $N^*(pt)$. This element of $N^*(pt)$ is determined by the Conner-Floyd characteristic classes of TX and f^*TY, and the type Σ.

□

<u>Example.</u>

Let $S^n \to \mathbb{R}^m$ be a map, where S^n is the n-sphere. S^n and $\mathbb{R}^m$ have trivial Conner-Floyd characteristic classes. Therefore, if the map satisfies the hypothesis of the theorem for any resolved singularity type then the cobordism class representing the particular singularity manifold is zero. In particular, the singularity manifold is the boundary of some compact manifold.

§ 7 The desingularisation of Σ^i.

We now consider the desingularisation of the first order Thom-Boardman singularities given by Porteous [22] and Ronga [25]. We use the word desingularisation of Σ^i since we have stronger properties than that required for a resolution of Σ^i as demanded by Theorem 17. Porteous and Ronga set up the geometry to compute the $\mathbb{Z}_2$ cohomology Thom polynomials for Σ^i. Passage to $\mathbb{Z}_2$ cohomology loses some information. Here we use their geometrical constructions and work with unoriented cobordism theory. In this context it is convenient to consider the sigularities of a vector bundle morphism. That is, let α^a and β^b be vector bundles over a manifold X with rank a and b respectively. We construct the bundle $HOM(\alpha,\beta) \to X$, a vector bundle morphism is a section of this bundle. If $f: X \to Y$ is a map between two manifolds X and Y then f induces a section of the bundle $HOM(TX, f^*TY) \to X$.

A fibre of the bundle $HOM(\alpha,\beta)$ consists of linear maps from $\mathbb{R}^a$ to $\mathbb{R}^b$. This fibre can be stratified by rank.

$$\Sigma^i = \{ g \in L(\mathbb{R}^a, \mathbb{R}^b) \mid \text{rank } g = a-i \}.$$

This is the first order Thom-Boardman stratification.

We adopt the same symbol Σ^i for the induced subbundles of $HOM(\alpha, \beta)$. In this chapter a generic vector bundle morphism by definition induces a section which is transverse to all the Σ^i.

Let α^a, β^b be vector bundles over X. Ronga ([25] Proposition 1.1) gives the following desingularisation of Σ^i:

$$\begin{array}{ccccc}
HOM(\gamma^i_{p^*\alpha}, \overline{\Pi}^* p^* \beta) & & & & \\
\downarrow & & & & \\
G_i(p^*\alpha) & \xrightarrow{\overline{\Pi}} & HOM(\alpha,\beta) & \hookleftarrow & \Sigma^i \\
 & & \downarrow p & & \\
 & & X & &
\end{array}$$

§7 The desingularisation of Σ^i.

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$$\begin{tikzcd}
HOM(\gamma^i_{p^*\alpha}, \bar{\Pi}^* p^* \beta) \arrow[d] & & \\
G_i(p^*\alpha) \arrow[r, "\bar{\Pi}"] & HOM(\alpha,\beta) \arrow[d, "p"] & \Sigma^i \arrow[l, hook'] \\
& X &
\end{tikzcd}$$

Let γ^i be the canonical bundle over the Grassmannian and $\overline{\Pi}$ the projection of the Grassmannian onto its base space. Ronga constructs a section of $HOM(\gamma^i_{p^*\alpha}, \overline{\Pi}^* p^*\beta)$ transverse to the zero section, whose zeros map under $\overline{\Pi}!$ onto the closure of the subfibre bundle Σ^i. A point of $G_i(p^*\alpha)$ is represented by (b,L), a linear map $L: \alpha_x \to \beta_x$ and an i-plane b of α_x, Ronga's section of $HOM(\gamma^i_{p^*\alpha}, \overline{\Pi}^* p^*\beta)$ is obtained by considering the restriction of L to the i-plane b. Thus the desingularisation of Σ^i as an element of $N^*(HOM(\alpha,\beta))$ is $\overline{\Pi}!\, e(\gamma^i_{p^*\alpha} \otimes \overline{\Pi}^* p^* \beta)$, where as throughout this chapter we identify a vector bundle with its dual.

We note that if $X = G_n \times G_m$ and α, β are the pullbacks over $G_n \times G_m$, of the canonical bundles over G_n and G_m we obtain a resolution of Σ^i as required for Theorem 17. However, because we have an explicit resolution respecting the stratification $\{\Sigma^j\}$ we have more information. We first prove a lemma.

<u>Lemma 13.</u>

Let α, β and δ be vector bundles over X and construct the Grassmannians $G_k(\alpha)$ and $G_k(\alpha \oplus \delta)$

$$\begin{tikzcd} G_k(\alpha) \arrow[r, "i"] \arrow[dr, "\Pi_\alpha"'] & G_k(\alpha \oplus \delta) \arrow[d, "\Pi_{\alpha+\delta}"] \\ & X \end{tikzcd}$$

When $G_k(\alpha)$ is considered as an element of $N^*(G_k(\alpha \oplus \delta))$ by the inclusion i, we have,

$$[\, G_k(\alpha),\ i\,] = e(\gamma^k_{\alpha+\delta} \otimes \Pi^*_{\alpha+\delta}\, \delta)\ ,$$

and

$$\Pi_\alpha!\ e(\gamma_\alpha^k \otimes \Pi_\alpha^* \beta) = \Pi_{\alpha+\delta}!\ e(\gamma_{\alpha+\delta}^k \otimes \Pi_{\alpha+\delta}^* (\beta \oplus \delta)) .$$

Proof.

For the first part it suffices to show that $G_k(\alpha)$ is represented by the zeros of a section of $HOM(\gamma_{\alpha+\delta}^k, \Pi_{\alpha+\delta}^* \delta)$, which is transverse to the zero section. The bundle
$HOM(\gamma_{\alpha+\delta}^k, \Pi_{\alpha+\delta}^* \delta) \to G_k(\alpha \oplus \delta)$ has a natural section, each k-plane in $(\alpha \oplus \delta)_x$ is projected onto δ_x. This section is transverse to the zero sectio

The second part follows since,

$$\Pi_\alpha!\ e(\gamma_\alpha^k \otimes \Pi_\alpha^* \beta) = \Pi_{\alpha+\delta}!\ i!\ e(i^* \gamma_{\alpha+\delta}^k \otimes i^* \Pi_{\alpha+\delta}^* \beta)$$

$$= \Pi_{\alpha+\delta}!\ e(\gamma_{\alpha+\delta}^k \otimes \Pi_{\alpha+\delta}^* \beta) .\ i!\ (1)$$

$$= \Pi_{\alpha+\delta}!\ e(\gamma_{\alpha+\delta}^k \otimes \Pi_{\alpha+\delta}^* (\beta \oplus \delta)) .$$

□

Theorem 18.

Let α^a, β^b be vector bundles over X and $s: \alpha \to \beta$ a generic vector bundle morphism. Then the closure of the singularities of type Σ^i is represented in $N^*(X)$ by a universal formula which is evaluated on the Conner-Floyd characteristic classes of $\beta - \alpha$, and depends only on i and $b-a$.

Proof.

Consider the following commutative diagram.

$$\begin{tikzcd}
\mathrm{HOM}(\gamma^i_{p^*\alpha}, \bar{\pi}^* p^*\beta) \arrow[r] & G_i(p^*\alpha) \arrow[r, "\bar{\pi}"] \arrow[d, "q"] & \mathrm{HOM}(\alpha,\beta) \arrow[d, "p"] \\
\mathrm{HOM}(\gamma^i_\alpha, \pi^*_\alpha \beta) \arrow[r] & G_i(\alpha) \arrow[r, "\pi_\alpha"] & X \arrow[u, bend right, "s"']
\end{tikzcd}$$

Let $\bar{\pi}$, π_α, p and q be the obvious projections. We consider the cobordism class in $N^*(\mathrm{HOM}(\alpha,\beta))$ representing the closure of $\Sigma^i(s)$.

$$S^* \bar{\pi}!\ e(\gamma^i_{p^*\alpha} \otimes \bar{\pi}^* p^* \beta) = p^{*-1} \bar{\pi}\ !\ e(\gamma^i_{p^*\alpha} \otimes \bar{\pi}^* p^* \beta)$$

$$= \pi_\alpha!\ q^{*-1} e(\gamma^i_{p^*\alpha} \otimes \bar{\pi}^* p^* \beta)$$

$$\text{since } \bar{\pi}!\ q^* = \pi_\alpha!\ p^*$$

$$= \pi_\alpha!\ e(\gamma^i_\alpha \otimes \pi^*_\alpha \beta).$$

By lemma 13 this is equal to

$$\pi_{\alpha+\delta}\ !\ e(\gamma^i_{\alpha+\delta} \otimes \pi_{\alpha+\delta}{}^* (\beta \otimes \delta))$$

so in particular if $\delta = -\alpha$ we obtain the required result.

□

So as in Ronga ([25] Theorem 4.5) we obtain the formula of Porteous ([22] Proposition 1.3) for the desingularisation of Σ^i. We obtain the same formula but all the symbols have a new interpretation, that of unoriented cobordism theory.

Ronga [25] has also given a desingularisation of the second order Thom-Boardman types $\Sigma^{i,j}$, so this gives rise to a result corresponding to Theorem 18. Other computations of Thom polynomials such as Porteous [22],

[23] also give geometrical constructions which can be evaluated using unoriented cobordism theory to obtain more information, than that obtainable by using Z_2 cohomology. In fact, we even lose information by using cobordism theory. The singularity set under consideration is usually represented by a manifold of zeros of a section transverse to the zero section of some bundle. We could ask questions such as, can this manifold be non-orientable or orientable or connected or a sphere?

These questions are answered by studying the zeros of sections of bundles that are transverse to the zero section. We do not consider these questions here.

Example.

Let β^n be a vector bundle over X and ε^{n-k+1} be the trivial n-k+1 plane bundle over X. The closure of the singularities of type Σ^1 for a generic map $\varepsilon^{n-k+1} \to \beta^n$ defines a cobordism class in X. The resolved manifold in $G_1(\varepsilon^{n-k+1})$ is the "obstruction" to finding n-k+1 linearly independent crosssections of β. The resolved manifold in $G_1(\varepsilon^{n-k+1})$ maps to a manifold with "singularities" in X, which when considered as an element in the Z_2 cohomology of X is the k-th Stiefel-Whitney class of the bundle β. In the case of cobordism theory this geometric characteristic class does not reduce to the k-th Conner-Floyd characteristic class, but is by definition $\sigma_k(\beta^n)$, the k-th McCrory characteristic class [17] .

That is, the McCrory characteristic class

$$\sigma_k(\beta) = \Pi_\varepsilon !\, e(\gamma_\varepsilon^1 \otimes \Pi_\varepsilon^* \beta^n).$$

Let $[\, p^a \,]$ be the cobordism operation that sends the pair $[\, M,f \,]$

to the pair $[M\times P^a\ ,\ f\circ \mathrm{proj}]$, where P^a is the a-dimensional projective space. We have the following lemma.

Lemma 14 (McCrory [17]).

(a) $\sigma_k(\beta^n) = \sum_{q=k}^{n} [P^{q-k}]\, \hat{w}_q(\beta^n)$

(b) This characteristic class gives rise to a stable cobordism operation.

$$\begin{aligned} \Theta^k : N^p(Y) &\to N^{p+k}(Y) \\ [N,g] &\to g!\sum_{q=k}^{n} [P^{q-k}]\ \hat{w}_q(g^*TY-TN) \\ &\qquad \| \\ &\quad\ g!\sum_{q=k}^{n} [P^{q-k}]\, R^q\, [N,g]\ . \end{aligned}$$

Proof.

This is McCrory's $\Theta = PR$ theorem. See [17].

□

Example.

We consider the singularities of a generic map $P^4 \to \mathbb{R}^3$, where P^4 is 4-dimensional projective space. The singular set is a 2-dimensional manifold. The map $P^4 \to \mathbb{R}^3$ induces a generic section s of the bundle $\mathrm{HOM}(TP^4,\ \varepsilon^3) \to P^4$. Since there is an isomorphism with the bundle $\mathrm{HOM}(\varepsilon^3,\ TP^4)$, we have a section s^* induced by transposition, which is also generic. The 2-dimensional singular manifold is also the singularity set for this induced section. So it suffices to consider the vector bundle morphism $\varepsilon^3 \to TP^4$. The total singularity set is then given by the McCrory characteristic class $\sigma_2(TP^4)$, which by Lemma 14 is $\hat{w}_2(P^4) + [P^2]\,.\,\hat{w}_4(P^4)$.

The top tom Dieck class $\hat{w}_4(P^4) \in N^4P^4 \cong Z_2$ maps by the epimorphism μ: $N^4P^4 \to H^4P^4$ onto the top Stiefel-Whitney class of P^4, which is non-zero, since P^4 has odd Euler characteristic. So $\hat{w}_4(P^4)$ is represented by a point. To compute the second tom Dieck class we consider the construction given by Milnor [19, Theorem 4.5]. That is,

$$TP^4 \oplus \varepsilon = \gamma^1 \oplus \gamma^1 \oplus \gamma^1 \oplus \gamma^1 \oplus \gamma^1 ,$$

where ε is the trivial line bundle over P^4 and γ^1 the canonical line bundle over P^4. From this it follows that $\hat{W}(P^4)= \hat{W}(\gamma^1)^5$. In particular,

$$\hat{w}_2(P^4)=\Sigma\ \hat{w}_i(\gamma^1)\hat{w}_j(\gamma^1)\hat{w}_k(\gamma^1)\hat{w}_\ell(\gamma^1)\hat{w}_m(\gamma^1),$$

summed over all i,j,k,ℓ,m such that $i+j+k+\ell+m = 2$. Since $\hat{w}_q(\gamma^1)= 0$ for $q > 1$ we have

$$\hat{w}_2(P^4)= \binom{5}{2} \hat{w}_1^2 (\gamma^1)\hat{w}_0^3(\gamma^1) + 2 \binom{5}{3} \hat{w}_1^3(\gamma^1)\hat{w}_0(\gamma^1)\hat{w}_{-1}(\gamma^1)$$

$$+ \binom{5}{4} \hat{w}_1^4(\gamma^1)\ \hat{w}_{-2}(\gamma^1).$$

This reduces to $\hat{w}_1^4(\gamma^1)\ \hat{w}_{-2}(\gamma^1) = e^4(\gamma^1).\ \hat{w}_{-2}(\gamma^1)$

where

$$\hat{w}_{-2}(\gamma^1) = \sum_i \eta_{i3}\ e(\gamma^1)^i ,$$

using the Formal group law.

Since $e^5(\gamma^1) = 0$ it suffices to consider the term η_{03}, but from the properties of a Formal group law (see [24]) it follows that η_{03} is zero.

So we have shown that $\hat{w}_2(P^4)= 0$ and $\hat{w}_4(P^4)$ is represented by a point. Therefore the singular manifold is cobordant to $[\ P^2,\ \text{const}\ ]$. In particular it is non-orientable and of odd Euler characteristic. The cohomology Thom polynomial for Σ^2, see Porteous [22] , is zero, since $w_2(P^4)= 0$ ($\hat{w}_2(P^4)$ maps onto $w_2(P^4)$). Therefore the singularity is not even detected by the cohomology Thom polynomial.

§8 Applications to parametrised maps.

We now apply the results of unoriented cobordism theory to study good maps $F: X \times C \to \mathbb{R}$. The following two theorems correspond to Theorem 2 and Theorem 5 .

Theorem 19.

Let $F: X \times C \to \mathbb{R}$ be a good map, then the cobordism class representing the critical manifold M is

$$\check{e}(X)[* \times C, \text{ inclusion }] \in N^*(X \times C).$$

Where $\check{e}(X)$ is the mod 2 integer representing the Euler class of X.

Proof.

Consider the following commutative diagram.

$$\begin{tikzcd} X \times C \arrow[r, "\tilde{F}"] \arrow[dr, "p"] & J^1(X) = T^*X \arrow[d, shift right, "\pi"'] \\ M \arrow[u, "i"] \arrow[r, "\tilde{F}|_M"] & X \arrow[u, shift right, "j"'] \end{tikzcd}$$

Where i and j are inclusions and p and π are projections. $\tilde{F}$ is transverse to X in T^*X and $M = \tilde{F}^{-1}(X)$, therefore

$$\begin{aligned} [M, i] &= \tilde{F}^* [X, j] \\ &= p^* \pi^{*-1} [X, j] \\ &= p^* j^* [X, j] \\ &= p^* e(X) \\ &= \check{e}(X)[* \times C, \text{ incl }]. \end{aligned}$$

□

Examples.

1 Let $F\colon \mathbb{C}P^2 \times P^2 \to \mathbb{R}$ be a good map where $\mathbb{C}P^2$ is the complex projective plane and P^2 the real projective plane. Since $e(\mathbb{C}P^2) \neq 0$, the inclusion of the critical manifold M^2 in $\mathbb{C}P^2 \times P^2$ is cobordant to a copy of the parameter space P^2. In particular M is non-orientable and has odd Euler characteristic.

2 Let $F\colon S^6 \times P^4 \to \mathbb{R}$ be a good map where S^6 is the 6-sphere and P^4 the 4-dimensional projective space. Since $e(S^6)=0$ the inclusion of the critical manifold M^4 is cobordant to zero in $S^6 \times P^4$. In particular it is the boundary of some compact 5-manifold.

As in Theorem 5, if the state space X is 1-dimensional all the singularity sets are manifolds. For the theorem below we use the notation of Theorem 5. M_q are the cuspoids of codimension q in $X^1 \times C$.

Theorem 20.

Let $F\colon X^1 \times C \to \mathbb{R}$ be a generic map. Then for all $q > 0$, the cobordism class representing the manifold M_q is zero in the cobordism of M_{q-1} .

Proof.

Consider the following commutative diagram as in the proof of Theorem 5.

$$\begin{tikzcd}
M_{q-1} \arrow[r, "h"] & I^k_{q-1}(X) \\
M_q \arrow[u, "j"] \arrow[r, "\tilde{h}"] & I^k_q(X) \arrow[u, "\ell"']
\end{tikzcd}$$

Instead of Lemma 1 we have

$$[\,M_q,\ j\,] = h^* \,[I^k_q(X),\ \ell\,] \quad .$$

The following diagram homotopy commutes, where p is the restriction of the projection $X \times C \to X$.

$$\begin{tikzcd} M_{q-1} \arrow[r, "h"] \arrow[dr, "p"'] & I^k_{q-1}(X) \\ & X \arrow[u, "s_{q-1}"'] \end{tikzcd}$$

We have

$$[\, M_q, j \,] = p^* s_{q-1}{}^* [\, I^k_q(X), \ell \,]$$

$$= 0$$

Since s_{q-1} avoids $I^k_q(X)$.

$\square$

<u>Example.</u>

Let $F: S^1 \times S^3 \to \mathbb{R}$ be a generic map, where S^1 and S^3 denote the circle and 3-sphere respectively. The fold manifold is cobordant to zero in the critical manifold M^3, in particular it is the boundary of a compact 3-manifold.

We now consider other singularity sets which are manifolds.

<u>Theorem 21.</u>

Let $F: X \times C \to \mathbb{R}$ be a good map with $\hat{F}$ transverse to the type $\Sigma^{n,k}$. If the closure of the singularities of type $\Sigma^{n,k}$ forms a manifold then the cobordism class representing the inclusion of this manifold in the cobordism of the critical manifold M is zero.

<u>Proof.</u>

Consider the following diagram.

$$\begin{array}{ccccccc} HOM(\gamma^k \circ i^*p^*TX, \mathbb{R}) & & i^*p^*TX & & & & \\ \downarrow & & \downarrow & & & & \\ G_k(i^*p^*TX) & \xrightarrow{\Pi_k} & M & \xrightarrow{i} & X\times C & \xrightarrow{p} & X \end{array}$$

The second derivative defines a section of $HOM(i^*p^*TX \circ i^*p^*TX, \mathbb{R}) \to M$. This induces a section of $HOM(\gamma^k \circ i^*p^*TX, \mathbb{R})$, transverse to the zero section, which is zero only above a singularity of type $\Sigma^{n,k}$. Therefore, the singularities of type $\Sigma^{n,k}$ in M are represented by the zeros of a section of the bundle $HOM(\gamma^k \circ i^*p^*TX, \mathbb{R})$ which is transverse to the zero section, so this defines the cobordism class independent of F. Since the cobordism class is independent of F, we can choose F to be the suspension of a Morse function on X. In this case there are no singularities of type $\Sigma^{n,k}$ and hence the cobordism class is zero.

□

Example.

Consider a locally stable map $F: S^8 \times S^5 \to \mathbb{R}$, where S^8 and S^5 denote the 8-sphere and 5-sphere respectively. Then the umbilic manifold of dimension two (corresponding to the singularities of type $\Sigma^{8,2}$) is the boundary of some compact 3-manifold.

§9 The image of the singular set.

We now investigate the image of the singular set, we have the following theorem.

Theorem 22.

Let $f: X^n \to Y^{n+r}$ with $r \geqslant 0$ be a proper map, whose induced section of f^*TY-TX is transverse to all the first order Thom-Boardman singularities. Then the cobordism class representing the total singular set is zero in $N^{2r+1}(Y)$.

Proof.

The closure of the singularities of type Σ^1 is represented in $N^*(X)$ by the McCrory characteristic class $\sigma_{r+1}(f^*TY-TX)$. This stable characteristic class gives rise to a cobordism operation Θ^{r+1} as in Lemma 14.

$$\Theta^{r+1} : N^r(Y) \to N^{2r+1}(Y)$$
$$[X,f] \to f!\,\sigma_{r+1}(f^*TY-TX)$$

By Lemma 14 $\Theta^{r+1}[X,f] = \sum_{q=r+1} \left[p^{q-(r+1)} \right] R^q\,[X,f]$.

This is zero, since $R^q[X,f] = 0$ for $q > r$.

□

Corollary 1.

Let X and Y be compact manifolds and $f: X^n \to Y^{n+r}$, $r \geqslant 0$ be a generic map as in the theorem. If the closure of the singularities of type Σ^1 is a manifold then this manifold is the boundary of a compact manifold.

□

Example.

Let X^3 and Y^3 be compact manifolds, then for an open dense set maps, there is a manifold M^2 of singularities of type Σ^1.

This manifold is the boundary of a compact 3-manifold. In particular the projective plane cannot occur as the total singular manifold.

Example.

Let $f\colon P^7 \to \mathbb{R}^9$ be a generic map, where P^7 is 7-dimensional projective space. The singular manifold has dimension 4 and is the boundary of a compact 5-dimensional manifold. This follows from the corollary, since the map f can easily be extended to a map $P^7 \to S^9$, where S^9 is the 9-sphere, which is compact.

Without the condition $r \geq 0$, the theorem would be false. For example, a generic function from the projective plane to the circle has, by the Thom polynomial theorem, an odd number of critical points. The result of the corollary is false if X and Y are not compact and in fact all manifolds can occur as the total singular set. Let M be a manifold and define a map

$$\begin{aligned} M \times \mathbb{R} &\longrightarrow M \times \mathbb{R} \\ (m,x) &\longmapsto (m,x^2), \end{aligned}$$

then M is the total singular set.

We now restrict attention to cohomology where we consider the Steenrod operations. Recall the geometric definition of Sq^i.

$$Sq^i : H^k(Y) \to H^{k+i}(Y)$$

$$\mu[M,g] \to g_! w_i(\nu_g) \quad \text{where} \quad \nu_g = g^* TY - TM .$$

A definition in terms of characteristic classes can also be given for the iterated Steenrod squares using the Riemann-Roch theorem.

<u>Lemma 15.</u>

$$Sq^p Sq^1 : H^k(Y) \to H^{k+(p+1)}(Y)$$

is the map $\quad [M,g] \to g!\left[w_p(v_g)\, w_1(v_g) + w_{p-1}(v_g)\, w_1^2(v_g) \right]$

<u>Proof.</u>

$$Sq^p Sq^1 \; [M,g] = Sq^p g!\, w_1(v_g)$$

$$= g!\left[\sum_{i+j=p} w_i(v_g).Sq^j(w_1(v_g)) \right]$$

by the Riemann-Roch Theorem.

$$= g!\left[w_p(v_g)\, Sq^0(w_1(v_g)) + w_{p-1}(v_g) Sq^1(w_1(v_g)) \right].$$

$$= g!\left[w_p(v_g)\, w_1(v_g) + w_{p-1}(v_g)\, w_1^2(v_g) \right].$$

□

We now illustrate this result by considering some Thom polynomials in $\mathbb{Z}_2$ cohomology.

<u>Example.</u>

Let X^3, Y^3 be compact and $f: X^3 \to Y^3$ type transverse to the Thom-Boardman type $\Sigma^{1,1,1}$. Then the number of singularities of type $\Sigma^{1,1,1}$ is even.

<u>Proof.</u>

The Thom polynomial for the singularities of type $\Sigma^{1,1,1}$ was computed by Porteous [22] as $c_2c_1 + c_1^3$ where c_i is the Stiefel-Whitney class of the difference bundle f^*TY-TX. By Lemma 15 we have that the image in Y is given by

$$Sq^2\, Sq^1 : H^0(Y) \to H^3(Y)$$

$$[X,f] \to f!(c_2c_1+c_1^3).$$

$Sq^2 Sq^1$ is zero on $H^0(Y)$ so the number of singularities of type $\Sigma^{1,1,1}$ must be even.

Other examples using $\mathbb{Z}_2$ cohomology can be given if the Thom polynomials correspond to iterated Steenrod squares.

Example 1.

Let L^2 and C^2 be compact and $L \to T^*C$ a Lagrangian immersion, then generically the induced projection $\chi : L^2 \to C^2$ has isolated singularities of type $\Sigma^{1,1}$, (See Arnold [2]). The Thom polynomial for $\Sigma^{1,1}$ is given by Porteous [22] as $c_1^2 + c_2$, where c_i denotes the Stiefel-Whitney class of the difference bundle $\chi^*TC\text{-}TL$. By Theorem 16 $c_1^2 = 0$, so in our case the Thom polynomial reduces to c_2, which gives rise to the Steenrod operation $Sq^2 : H^0(C) \to H^2(C)$. This is zero and so the number of singularities of type $\Sigma^{1,1}$ is even.

Example 2.

Let L^4 and C^4 be compact and $L \to T^*C$ a Lagrangian immersion, then generically the induced projection $\chi : L^4 \to C^4$ has isolated singularities of type $\Sigma^{1,1,1,1}$, (See Arnold [2]). We claim that this number is always even. The Thom polynomial is computed by Sergeraert [26] as $c_1^4 + c_1c_3$, where c_i denotes the Stiefel-Whitney class of the difference bundle $\chi^*TC\text{-}TL$. In our case by Theorem 16 $c_1^4 + c_1c_3 = c_1^2\, c_2 + c_1c_3$. But this characteristic class gives rise to the Steenrod operation $Sq^3\, Sq^1$, which is zero $H^0(C)$. This establishes the claim.

that the image in Y is given by

$$Sq^2\, Sq^1 : H^o(Y) \to H^3(Y)$$

$$[X,f] \to f!(c_2c_1+c_1^3).$$

$Sq^2\, Sq^1$ is zero on $H^o(Y)$ so the number of singularities of type $\Sigma^{1,1,1}$ must be even.

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Appendix.

A Lagrangian Klein bottle.

By J. Hayden & E. C. Zeeman.

Lagrangian theory (2) and elementary catastrophe theory (5) coincide locally but differ globally. Locally the singularities that occur in the two theories are the same, but globally the manifolds that occur are different. We shall show that Lagrangian manifolds are more general.

By a catastrophe manifold M over a parameter space C we mean a component of the critical set of a family F of functions parametrised by C. It is well-known (e.g. 6, 7) that F induces an immersion of M as a Lagrangian submanifold of the cotangent bundle T^*C of C. It is also well-known that not every Lagrangian immersion can be induced in this way; indeed the Maslov index (1) is an obstruction. More subtle is the fact that there exist manifolds that can be immersed as Lagrangian submanifolds, but which cannot even occur as catastrophe manifolds. It is valuable to have an explicit low dimensional example, because surprisingly few Lagrangian manifolds are known (other than those induced from functions).

The simplest candidate for such a manifold must be the Klein bottle K due to the constraints on the Stiefel-Whitney classes imposed by the obstruction theories of Hayden (4). In this paper we verify that K is indeed such a manifold by constructing an explicit immersion of K as a Lagrangian submanifold of $T^*\mathbb{R}^2$. Meanwhile the non-orientability of K prevents it from occuring as a catastrophe manifold over $\mathbb{R}^2$.

Elementary catastrophe theory. Let $F: C \times X \to \mathbb{R}$ be a C^∞-function on a manifold X, parametrised by a manifold C. We assume (generically) that 0 is a regular value of $\frac{\partial F}{\partial x}$, and hence the critical set of F given by $\frac{\partial F}{\partial x} = 0$ is a smooth submanifold of $C \times X$ of the same dimension as C. Let M_F be a component of this critical set; we call M_F a catastrophe manifold. Let $\chi: M_F \to C$ be induced by projection.

$$
\begin{tikzcd}
M_F \arrow[rr, "\subset"] \arrow[dr, "\chi"'] & & C \times X \arrow[dl, "\text{proj}"] \\
& C &
\end{tikzcd}
$$

Let $W = \Sigma w_i$ denote the total Stiefel-Whitney class.

Theorem 1. (Hayden (4)). $W(M_F) = \chi^* W(C)$.

Corollary. If C is orientable so is M_F.
For C orientable $\quad w_1(C) = 0 \Longrightarrow w_1(M_F) = 0 \Longrightarrow M_F$ orientable.
Hence the Klein bottle K cannot occur as a catastrophe manifold over $\mathbb{R}^2$.

Lagrangian theory. The cotangent bundle T^*C has a natural symplectic structure ω. If M is a manifold of the same dimension as C, then an immersion $\varphi: M \to T^*C$ is called Lagrangian if ω vanishes on the tangent planes of φM. Let $\chi: M \to C$ be induced by projection.

$$
\begin{tikzcd}
M \arrow[rr, "\varphi"] \arrow[dr, "\chi"'] & & T^*C \arrow[dl, "\text{proj}"] \\
& C &
\end{tikzcd}
$$

Theorem 2 (Hayden (4)). $W^2(M) = \chi^* W^2(C)$.

The relationship between the two theorems is as follows. Given F, let $\varphi_F: C \times X \to T^*C$ be given by $\varphi_F(c,x) = (c, \frac{\partial F}{\partial c})$. Then $\varphi_F | M_F$ is a Lagrangian immersion (6, p. 716)

<u>Elementary catastrophe theory.</u> Let $F:C\times X \to \mathbb{R}$ be a C^∞-function on a manifold X, parametrised by a manifold C. We assume (generically) that 0 is a regular value of $\frac{\partial F}{\partial x}$, and hence the critical set of F given by $\frac{\partial F}{\partial x} = 0$ is a smooth submanifold of $C\times X$ of the same dimension as C. Let M_F be a component of this critical set; we call M_F a <u>catastrophe manifold</u>. Let $\chi:M_F \to C$ be induced by projection.

$$\begin{tikzcd} M_F \arrow[rr, "\subset"] \arrow[dr, "\chi"'] & & C\times X \arrow[dl, "\mathrm{proj}"] \\ & C & \end{tikzcd}$$

Let $W = \Sigma w_i$ denote the total Stiefel-Whitney class.

<u>Theorem 1.</u> (Hayden (4)). $W(M_F) = \chi^* W(C)$.

<u>Corollary.</u> If C is orientable so is M_F.
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<u>Lagrangian theory.</u> The cotangent bundle T^*C has a natural symplectic structure ω. If M is a manifold of the same dimension as C, then an immersion $\varphi:M \to T^*C$ is called <u>Lagrangian</u> if ω vanishes on the tangent planes of φM. Let $\chi:M \to C$ be induced by projection.

$$\begin{tikzcd} M \arrow[rr, "\varphi"] \arrow[dr, "\chi"'] & & T^*C \arrow[dl, "\mathrm{proj}"] \\ & C & \end{tikzcd}$$

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The relationship between the two theorems is as follows. Given F, let $\varphi_F:C\times X \to T^*C$ be given by $\varphi_F(c,x) = (c, \frac{\partial F}{\partial c})$. Then $\varphi_F | M_F$ is a Lagrangian immersion (6, p. 716)

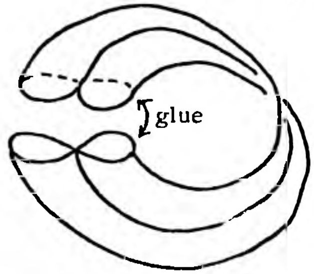
$$
\begin{tikzcd}
M_F \arrow[r, "\subset"] \arrow[dr, "\chi"'] & C \times X \arrow[r, "\varphi_F"] \arrow[d, "\mathrm{proj}"] & T^*C \arrow[dl, "\mathrm{proj}"] \\
& C &
\end{tikzcd}
$$

Hence in this special case Theorem 2 is a corollary of Theorem 1. However, in general Theorem 2 cannot be sharpened to Theorem 1. To prove this we choose C with W(C) trivial, and seek M such that $W^2(M)$ is trivial but W(M) is non-trivial. Hence M cannot be 1-dimensional, and if it is compact and 2-dimensional then it must be non-orientable with even genus. Hence the Klein bottle K is the simplest candidate. We now set about constructing a Lagrangian immersion of K in $T^*\mathbb{R}^2$, thus proving that the two theories differ.

Construction of the immersion.

We shall use an immersion in the form of a figure-eight multiplied by an interval, with the ends glued together by a half-rotation. Notice that a half-rotation of a figure-eight reverses orientation, and so this is indeed an immersed Klein bottle. Analytic immersions of this type do exist in $\mathbb{R}^3$ (3), but we shall need the extra room of the fourth dimension in order to satisfy the Lagrangian condition.

Figure 1.

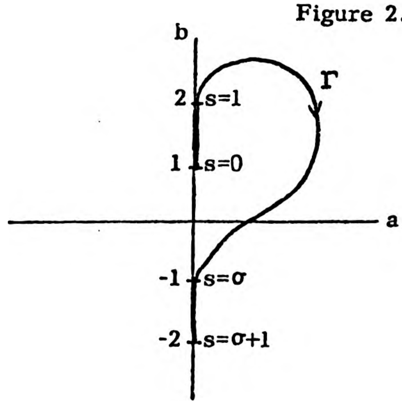


Let (a, b) be coordinates for $\mathbb{R}^2$, inducing coordinates (a, b, c, d) for $T^*\mathbb{R}^2$. The symplectic structure is then given by

$$\omega(u,v) = \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \begin{vmatrix} u_2 & u_4 \\ v_2 & v_4 \end{vmatrix}$$

where $u = (u_1, u_2, u_3, u_4)$, $v = (v_1, v_2, v_3, v_4)$ are two tangents at a point of $T^*\mathbb{R}^2$.

Figure 2.



Let Γ be a C^∞-curve in $\mathbb{R}^2$ that starts at (0, 1), runs up the b-axis to (0, 2), bends smoothly round to rejoin the b-axis at (0, -1), before finally descending down the b-axis to (0, -2), as shown in Figure 2. Let $\sigma+1$ be the length of Γ. Parametrise Γ by arc-length s, $0 \le s \le \sigma+1$, and, and let $(\alpha(s), \beta(s))$ be the coordinates of the point s. In particular, denoting the derivative with respect to s by a dash, we have :

$$(*) \qquad 0 \le s \le 1 \quad \Rightarrow \quad \alpha = \alpha' = 0,\ \beta = s+1,\ \beta' = 1.$$
$$\sigma \le s \le \sigma+1 \quad \Rightarrow \quad \alpha = \alpha' = 0,\ \beta = \sigma-s-1,\ \beta' = -1.$$

Choose ε, $0 < \varepsilon < (\max_s\{|\alpha''|^2 + |\beta''|^2\})^{-\frac{1}{2}}$. This ensures that ε is less than the radius of curvature everywhere, and so the ε-disks normal to Γ in $\mathbb{R}^2$ are locally disjoint; by further choosing ε sufficiently small we can ensure that they are also globally disjoint from one another.

Let $\zeta(a,b)$ denote the distance from (a,b) to Γ in $\mathbb{R}^2$. Let $\mathbb{R}^3 = \mathbb{R}^2 x\mathbb{R}$, with coordinates (a,b,x), and identify $\mathbb{R}^2 = \mathbb{R}^2 x0$. Let $F:\mathbb{R}^2 x\mathbb{R} \to \mathbb{R}$ be given by

$$F(a,b,x) = \frac{x^3}{3} + x(\zeta^2-\varepsilon^2) .$$

Then F has catastrophe manifold M_F given by

$$\frac{\partial F}{\partial x} = x^2 + \zeta^2 - \varepsilon^2 = 0.$$

Since M_F is the boundary of the ε-neighbourhood of Γ in $\mathbb{R}^3$, we can think of M as a tubular cylinder of radius ε running along Γ, together with two hemispheres at the ends. We shall in fact only use the cylinder part, which we call A. The cylinder is the union of the circles bounding the ε-disks normal to Γ in $\mathbb{R}^3$, which are disjoint by our choice of ε. Each circle can be parametrised by θ, such that

$$\zeta = |\varepsilon \cos\theta|, \qquad x = \varepsilon \sin\theta.$$

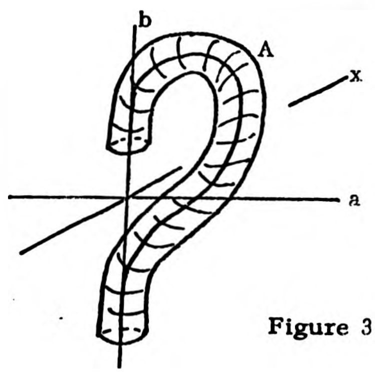


Figure 3.

Hence A is parametrised by (s,θ), $0 \le s \le \sigma+1$, $0 \le \theta < 2\pi$, and the coordinates of the point (s,θ) in $\mathbb{R}^3$ are given by

$$(a,b,x) = (\alpha+\varepsilon\beta'\cos\theta, \beta-\varepsilon\alpha'\cos\theta, \varepsilon\sin\theta).$$

Now F induces a Lagrangian immersion of M, and hence of A in $T^*\mathbb{R}^2$, given by

$$(a,b,x) \longmapsto (a,b,\frac{\partial F}{\partial a},\frac{\partial F}{\partial b})$$

Since s measures arc-length, $\frac{\partial \zeta}{\partial a} = \beta'$, $\frac{\partial \zeta}{\partial b} = -\alpha'$. Therefore at (s,θ),

$$\frac{\partial F}{\partial a} = 2x\zeta\frac{\partial \zeta}{\partial a} = 2(\varepsilon \sin\theta)(\varepsilon\cos\theta)\beta' = \varepsilon^2\beta'\sin 2\theta$$

$$\frac{\partial F}{\partial b} = -\varepsilon^2\alpha'\sin 2\theta\ .$$

Therefore the Lagrangian immersion $\varphi_A : A \to T^*\mathbb{R}^2$ is given by

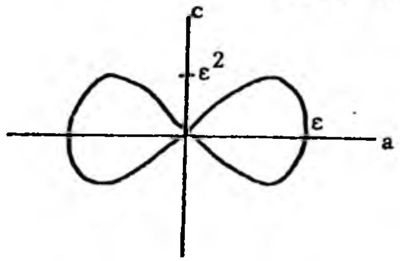
$$\varphi_A(s,\theta) = (\alpha+\varepsilon\beta'\cos\theta, \beta-\varepsilon\alpha'\cos\theta, \varepsilon^2\beta'\sin 2\theta, -\varepsilon^2\alpha'\sin 2\theta).$$

Notice that each of the circles s = constant is immersed as a figure-eight by φ_A; for example when $s = 0$, by (*),

$$\varphi_A(0,\theta) = (\varepsilon\cos\theta, 1, \varepsilon^2\sin 2\theta, 0).$$

This is a figure-eight as shown in Figure 4. Increasing s has the effect of isotoping this figure-eight in $T^*\mathbb{R}^2$ so that the self-intersection point runs along Γ, $\subset \mathbb{R}^2 \subset T^*\mathbb{R}^2$, and as it goes along the 2-plane containing the figure-eight rotates in the 3-planes normal to Γ.

Figure 4.



This rotation is just what is necessary to satisfy the Lagrangian condition.

We shall now construct the Klein Bottle by immersing another cylinder B and glueing it onto A; only this time we have to use a different process for manufacturing B in order to avoid getting a torus by mistake. Then the whole immersed Klein bottle will consist of a figure-eight running along the curve shown in Figure 5. Any 1-dimensional manifold immersed in $T^*\mathbb{R}$ is trivially Lagrangian, and the cartesian product of two Lagrangian manifolds is another; hence we can manufacture B by taking the product of the figure-eight shown in Figure 4 with a reflected copy Γ_1 of Γ in the (b, d)-plane, as shown in Figure 5. More precisely let B be the cylinder

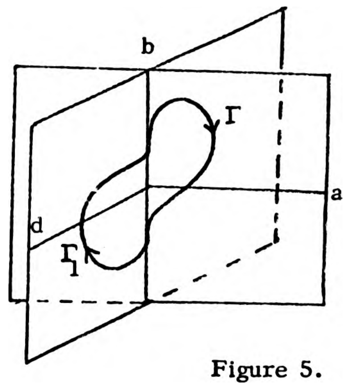


Figure 5.

$$B = \{(s, \theta);\ \sigma \le s \le 2\sigma+1,\ 0 \le \theta < 2\pi\} .$$

Let $\varphi_B : B \to T^*\mathbb{R}^2$ be the Lagrangian immersion given by

$$\varphi_B(s, \theta) = (-\varepsilon \cos\theta, -\beta(s-\sigma), -\varepsilon^2 \sin 2\theta, \alpha(s-\sigma)).$$

Note that we could have just written down the formulae for φ_A, φ_B and verified directly that they were Lagrangian immersions, by checking that the vectors $u = \frac{\partial\varphi}{\partial s}$, $v = \frac{\partial\varphi}{\partial\theta}$ were linearly independent and satisfied $\omega(u, v) = 0$. However this approach would have been less intuitive.

To glue the images of A, B together we define K abstractly, cover it with two charts, map the two charts by φ_A, φ_B, and verify that the maps agree on the overlap. Let S^1 be the circle parametrised by θ, $0 \le \theta < 2\pi$. define $\psi : \mathbb{R}\mathrm{x}S^1 \to \mathbb{R}\mathrm{x}S^1$ to be the diffeomorphism given by

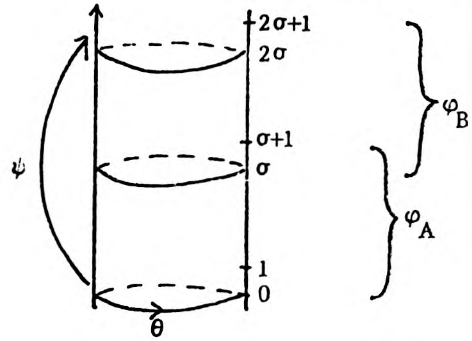
$$\psi(s, \theta) = (2\sigma+s, \pi-\theta) .$$

Let $K = \mathbb{R}\mathrm{x}S^1/\mathbb{Z}$, where $\mathbb{Z}$ is the integer group action generated by ψ. Then K is a Klein bottle because ψ translates $\mathbb{R}$ and reflects S^1.

Figure 6.

Define $\varphi: K \to T^*\mathbb{R}^2$ by

$$\varphi = \begin{cases} \varphi_A, & 0 < s < \sigma+1 \\ \varphi_B, & \sigma < s < 2\sigma+1 \end{cases}$$



Then charts overlap when $\sigma < s < \sigma+1$ and when $0 < s < 1$ (or equivalently $2\sigma < s < 2\sigma+1$ due to the identification under ψ), and so we have to verify that the definition is unambiguous in these cases. First, if $\sigma < s < \sigma+1$ then

$$\begin{aligned} \varphi_B(s,\theta) &= (-\varepsilon\cos\theta, -\beta(s-\sigma), -\varepsilon^2\sin 2\theta, \alpha(s-\sigma)) \\ &= (-\varepsilon\cos\theta, -(s-\sigma+1), -\varepsilon^2\sin 2\theta, 0), \quad \text{by } (*) \\ &= (-\varepsilon\cos\theta, \sigma-s-1, -\varepsilon^2\sin 2\theta, 0) \\ &= \varphi_A(s,\theta)\ , \ \text{by } (*)\ . \end{aligned}$$

Next, if $0 < s < 1$ then

$$\begin{aligned} \varphi_B\psi(s,\theta) &= \varphi_B(2\sigma+s, \pi-\theta) \\ &= (-\varepsilon\cos(\pi-\theta), -\beta(\sigma+s), -\varepsilon^2\sin 2(\pi-\theta), \alpha(\sigma+s)) \\ &= (\varepsilon\cos\theta, -(\sigma-(\sigma+s)-1), \varepsilon^2\sin 2\theta, 0) \\ &= (\varepsilon\cos\theta, s+1, \varepsilon^2\sin 2\theta, 0) \\ &= \varphi_A(s,\theta). \end{aligned}$$

Hence $\varphi_B\psi = \varphi_A$, and the definition is unambiguous as required. Finally φ is a C^∞ Lagrangian immersion since it has these properties on each open chart.

Remark. We do not know if the immersion can be made analytic or made into an embedding. Possibly w_1 is an obstruction to a Lagrangian embedding, as it is to an embedding in $\mathbb{R}^3$.

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